UNIVERSITY OF TORONTO SCARBOROUGH
Department of Computer and Mathematical Sciences
Midterm Test, March 2014

STAD57H3 Time Series Analysis
Duration: One hour and fifty minutes

Last Name:________________________ First Name: ________________________

Student number: ________________________

Aids allowed:

- The textbook (Time Series Analysis and Its Applications, with R examples; R.H. Shumway and D.S))
- Class notes
- A calculator (No phone calculators are allowed)

No other aids are allowed. For example you are not allowed to have any other textbook or past exams.

All your work must be presented clearly in order to get credit. Answer alone (even though correct) will only qualify for **ZERO** credit. Please show your work in the space provided; you may use the back of the pages, if necessary, but you MUST remain organized. Show your work and answer in the space provided, in ink. Pencil may be used, but then any re-grading will NOT be allowed.

There are 10 questions and 8 pages including this page. Please check to see you have all the pages and questions.

Good Luck!

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	12	8	10	8	11	3	3	3	3	4	65
Score:											

1. Consider the time series $X_t = 1 + 0.5t + W_t$ where $W_t \sim WN(0, \sigma^2)$ and let Y_t be the 5-point moving average defined by $Y_t = \frac{1}{5}\sum_{j=-2}^{2} X_{t+j}$.

   (a) (2 points) Give the mean function of X_t(i.e. $\mu_X(t) = E(X_t)$).

   Solution: $\mu_X(t) = E(X_t) = 1 + 0.5t + EW_t = 1 + 0.5t$ since $EW_t = 0 \quad \forall t$ ■

   (b) (3 points) Calculate the mean function of Y_t(i.e. $\mu_Y(t) = E(Y_t)$). Show your work clearly and give your answer in the simplest form. Eg You should evaluate all summations involved.

   Solution: $EY_t = \frac{1}{5}\sum_{j=-2}^{2} EX_{t+j} = \frac{1}{5}\sum_{j=-2}^{2}(1 + 0.5(t+j)) = 1 + 0.5t$ since $\sum_{j=-2}^{2} j = 0$ ■

   (c) (2 points) What is the value of $Cov(X_t, X_{t+1})$.

   Solution: $Cov(X_t, X_{t+1}) = Cov(W_t, W_{t+1}) = 0$ ■

   (d) (5 points) Calculate $Corr(Y_t, Y_{t+1})$.

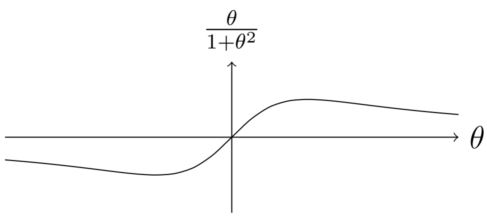
   Solution: $Cov(Y_t, Y_{t+1}) = \frac{1}{25}Cov(X_{t-2} + X_{t-1} + X_t + X_{t+1} + X_{t+2}, X_{t-1} + X_t + X_{t+1} + X_{t+2} + X_{t+3}) = \frac{4}{25}\sigma^2$

$$
\begin{aligned}
Var(Y_t) &= Var(\frac{1}{5}\sum_{j=-2}^{2} X_{t+j}) \\
&= \frac{1}{25}\sum_{j=-2}^{2} V(X_{t+j}) \\
&= \frac{1}{25}\sum_{j=-2}^{2} V(1 + 0.5(t+j) + W_{t+j}) \\
&= \frac{1}{25}\sum_{j=-2}^{2} V(W_{t+j}) \\
&= \frac{1}{25}5\sigma^2 \\
&= \frac{1}{5}\sigma^2.
\end{aligned}
$$

   $Corr(Y_t, Y_{t+1}) = \frac{Cov(Y_t, Y_{t+1})}{Var(Y_t)} = \frac{\frac{4}{25}\sigma^2}{\frac{1}{5}\sigma^2} = 0.8$ ■

2. (8 points) For an MA(1) process $X_t = W_t + \theta W_{t-1}$, show that $|\rho_X(1)| \leq 0.5$ for any θ. For which θ does $\rho_X(1)$ attain its maximum or minimum?

Solution: $\rho_X(1) = \frac{\theta}{1+\theta^2}$.
$\frac{d\rho_X(1)}{d\theta} = \frac{(1+\theta^2)-2\theta^2}{(1+\theta^2)^2} = \frac{1-\theta^2}{(1+\theta^2)^2}$.
$\frac{d\rho_X(1)}{d\theta} = 0 \implies \theta = \pm 1$.
When $\theta = 1$, $\rho_X(1) = 0.5$ (max) and $\theta = -1$, $\rho_X(-1) = -0.5$ (min) .



3. Consider the time series $X_t = \frac{1}{3}X_{t-1} + \frac{2}{9}X_{t-2} + W_t$ where $W_t \sim WN(0,1)$

   (a) (3 points) Show that this process is stationary?

   Solution: Show that both the roots of the AR characteristic equation lie outside the unit circle.

   The AR characteristic equation is $\phi(B) = 1 - \frac{1}{3}B - \frac{2}{9}B^2 = 0$ or $2B^2 + 3B - 9 = 0$. This has roots $\frac{-3\pm\sqrt{3^2+4\times2\times9}}{9} = \frac{-3\pm9}{4}$. i.e. -3 or $\frac{6}{4}$. Both the roots are greater than 1 and so the process is stationary.

   (b) (7 points) Calculate $\rho(2)$

   Solution: Solve Yule-Walker equations to get $\gamma(0)$ and $\gamma(1)$ and $\gamma(2)$ and $\rho(2) = \gamma(2)/\gamma(0)$.
   The Yule-Walter equations are $\gamma(h) - \sum_{i=1}^{p} \phi_i \gamma(h-i) = \sigma_w^2 I(h=0)$. We have $\phi_1 = \frac{1}{3}$, $\phi_2 = \frac{2}{9}$ and $\sigma_w^2 = 1$

   When $h = 0$, $\gamma(0) - \frac{1}{3}\gamma(1) - \frac{2}{9}\gamma(2) = 1$

   $h = 1$, $\gamma(1) - \frac{1}{3}\gamma(0) - \frac{2}{9}\gamma(1) = 0$

   $h = 2$, $\gamma(2) - \frac{1}{3}\gamma(1) - \frac{2}{9}\gamma(0) = 0$

   The second equation above gives $\gamma(1) = \frac{3}{7}\gamma(0)$ and substituting this in the third equation gives $\gamma(2) = \left(\frac{1}{7} + \frac{2}{9}\right)\gamma(0)$ or $\rho(2) = \frac{1}{7} + \frac{2}{9} = \frac{23}{63} = 0.365$ ■

4. (8 points) Consider the time series $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + W_t$ where $W_t \sim WN(0, \sigma^2)$. If this process is stationary, then prove that $|\phi_2| < 1$

Solution: Ch fn: $1 - \phi_1 B - \phi_2 B^2 = 0$. If r_1 and r_2 are the roots of this equation, then $|r_1 r_2| = 1/|\phi_2|$. Since the process is stationary , all roots must be are greater than 1 in modulus and this requires $1/|\phi_2| > 1$ i.e. $|\phi_2| < 1$ ■

5. (11 points) Consider the time series model $X_t = \phi^2 X_{t-2} + W_t + \theta W_{t-1}$, where $W_t \sim N(0, \sigma_w^2)$, $\phi = 0.5$ and $\theta = 0.8$. Calculate $\rho(1)$.

Solution: $X_t = (1 - \phi^2 B^2)^{-1}(1 + \theta B)W_t = (1 + \phi^2 B^2 + \phi^4 B^4 + \phi^6 B^6 + \cdots)(1 + \theta B)W_t = (W_t + \phi^2 W_{t-2} + \phi^4 W_{t-4} + \phi^6 W_{t-6} + \cdots) + (\theta W_{t-1} + \theta\phi^2 W_{t-3} + \theta\phi^4 W_{t-5} + \theta\phi^6 W_{t-7} + \cdots)$.
$\gamma(0) = Var(X_t) = \sigma_w^2 \frac{1}{1-\phi^4} + \theta^2 \sigma_w^2 \frac{1}{1-\phi^4} = \sigma_w^2 \frac{1+\theta^2}{1-\phi^4}$.
To calculate $\gamma(1)$ we rearrange the terms in X_t as linear process: $X_t = W_t + \theta W_{t-1} + \phi^2 W_{t-2} + \theta\phi^2 W_{t-3} + \phi^4 W_{t-4} + \theta\phi^4 W_{t-5} + \phi^6 W_{t-6} + \theta\phi^6 W_{t-7} + \cdots)$.
$\gamma(1) = \sigma_w^2 \sum_{k=0}^{\infty} \psi_k \psi_{k+1} = \sigma_w^2(\theta + \theta\phi^2 + \theta\phi^4 + \theta\phi^6 + \cdots) = \theta\sigma_w^2 \frac{1}{1-\phi^2}$.
$\rho(1) = \frac{\gamma(1)}{\gamma(0)} =$

This part consists of multiple choice questions. Just circle your answer. You do not have to show work in this part.

6. (3 points) Consider the model defined by $X_0 = 0$ and $X_t = X_{t-1} + W_t$, for $t = 1, 2, 3, \ldots$ where $W_t \sim N(0, \sigma_w^2)$. What is the value of $Corr(X_2, X_{50})$?

   A) 0

   B) 0.2

   C) 0.04

   D) 0.5

   E) 49

Solution: This is a random walk process. (p 12,13 my notes).
$Corr(X_s, X_t) = \sqrt{\frac{\min(s,t)}{\max(s,t)}}$
$Corr(X_2, X_{50}) = \sqrt{\frac{2}{50}} = 0.2$

7. (3 points) Which of the following processes is (are) non-stationary. In these processes, $W_t \sim WN(0, \sigma^2)$.

   A) $X_t = 0.5X_{t-1} + W_t - 1.5W_{t-1}$

   B) AR(2) process whose AR characteristic polynomial has roots $0.8 \pm 0.9i$.

   C) $(1 - 0.3B)X_t = (1 - B)W_t$

   D) $X_t = 2.25X_{t-1} - 0.5X_{t-2} + W_t$

   E) All the above processes are stationary.

Solution: a) White noise processes are stationary.
b) The absolute value of the root of the AR characteristic pol is $1/0.5 > 1$ and so stationary c) $|0.8 \pm 0.9i| > 1$ and so stationary. d) The absolute value of the root of the AR characteristic pol is $1/0.3 > 1$ and so stationary e) AR ch pol is $1 - 2.25B + 0.5B^2 = (1 - 2B)(1 - 0.25B)$ and one of the roots is $0.5 < 1$ and so not stationary.

8. (3 points) Consider the stationary AR(1) process with a constant term given by $X_t = 0.2 + 0.4X_{t-1} + W_t$ where $W_t \sim WN(0, \sigma^2)$. What is the value of $E(X_t)$ for this process?

   A) 0.2

   B) 0.33

   C) 0.4

Question 8 continues on the next page. . .

D) 0.5

E) For this process $E(X_t)$ depends on t.

Solution: $E(X_t) = \frac{0.2}{1-0.4} = 0.33$

9. (3 points) Consider the time series defined by $X_t = (1-B)Z_t$ where $Z_t = 1 + 2t + W_t$ and $W_t \sim WN(0, \sigma^2)$. If $\sigma^2 = 2$, what is the value of $Var(X_t)$?

   A) 1

   B) 2

   C) 3

   D) 4

   E) 6

Solution: $X_t = (1-B)Z_t = Z_t - Z_{t-1} = (1 + 2t + W_t) - (1 + 2(t-1) + W_{t-1}) = 2 + W_t - W_{t-1}$ and so $Var(X_t) = V(W_t) + V(W_{t-1}) = 2 + 2 = 4$

10. (4 points) Read the following statements regarding the two processes $U_t = W_t + 3W_{t-1}$ and $V_t = W_t + \frac{1}{3}W_{t-1}$ In these processes, $W_t \sim WN(0, \sigma^2)$.
State whether the following statements are true or false (Just circle your answer. One mark for each correct answer)

   i Both U_t and V_t have the same autocovariance function i.e. $\gamma_U(h) = \gamma_V(h) \quad \forall h$ (TRUE / FALSE)

   ii Both U_t and V_t have the same autocorrelation function i.e. $\rho_U(h) = \rho_V(h) \quad \forall h$ (TRUE / FALSE)

   iii Both U_t and V_t are stationary. (TRUE / FALSE)

   iv Both U_t and V_t are invertible. (TRUE / FALSE)

Solution: False,True, True, False

END OF TEST