LIFE ANNUITIES

5-1 Introduction:

Over the years the life insurance business has come to be one of the most important industries in the nation. According to a report made by the American Council of Life Insurance, the insurance companies of the United States had assets worth $771 million in 1890, but this amount had increased to more than $1,942 billion in 1994.' Most of these dollars are earmarked to meet the life insurance companies' future obligations to policy-holders. One such obligation is the life annuity. which will be discussed in this chapter.

The basic idea of a life annuity is simple. People generally find it rather difficult to save enough money during their working years to support themselves in their old age. Furthermore, individuals themselves would not need their savings should they die before reaching retirement age. With a life annuity, each person within an age group con-tributes an equal amount of money to an agent, such as an insurance company, for the purpose of sharing in the total amount at a future date, provided that person is still alive.

Survivors can later enjoy an amount larger than their original contributions, since some of the contributors will not be alive then and interest is paid by the insurance company for use of the contributions. Since no one knows if he or she will be alive on a particular future date and the annuity is payable depending on a future occurrence, a life annuity is a type of contingent annuity.

The basic idea of life insurance is the same as that of life annuity. Both life insurance and life annuity contracts are designed for protection against the contingency of future life. However, a life annuity is provided for the purchaser of the annuity (the annuitant) to use the money in old age, whereas the purchaser of life insurance (the insured) is not usually the one who receives the benefits of the life insurance contract (the policy). A life annuity is payable to the annuitant by the insurance company if the annuitant is still alive on an indicated date, according to the annuity contract. Life insurance is payable to the beneficiary of the insured if the insured dies within the indicated time stated in the insurance policy. Thus, the life annuity is not a life insurance contract. However, since the mathematics of a life annuity, or actuarial calculation, is based on a concept similar to that used in life insurance, life annuities have always been considered as part of the life insurance business and are among the oldest types of insurance contracts.

The cost, which is used to provide funds for the payment of the benefit according to a life annuity or an insurance contract, is called the net premium. In addition to the net premium, other amounts of money that are charged by an insurance company are called loading costs, which include the profit element and the operating expenses of the company, such as salaries, rent, and depreciation. The sum of the net premium and the loading costs is called the gross premium. Since both the operating expenses and the profit rates differ among insurance companies, for the sake of simplicity, only the net premium is computed in this and the following chapters

The fundamental relationship between the net premium and the future benefits under any type of annuity contract or insurance policy may be expressed on the purchase date as follows:

Present value of net premium = Present value of future benefit(s)

The net premium may be paid in a single amount, called the net single premium, on the purchase date. It may also be paid in equal annual payments, called the net annual premiums. Net single premiums are generally paid for annuity contracts, while net annual premiums are usually paid for life insurance policies. However, a knowledge of the net single premium is necessary in computing a set of net annual premiums for an insurance policy.

Insurance companies usually invest their collected premiums in various fields to earn interest at rates that fluctuate on the money market. The interest income is first used as a portion of the reserve for the benefit of policyholders. (See Section 20.7B, Uses of the Terminal Reserve.) If the interest income and the collected premiums exceed the reserve requirements, payments of claims, and administrative expenses, the insurance company may use the excess as a dividend payable to policyholders. In order to simplify the mathematical illustrations, unless otherwise specified, net premiums are computed at a nominal interest rate of 4% in all problems in this text.

Generally, the age of a purchaser on his or her last birthday (ALB) or nearest birthday (ANB) is taken in computing the purchase price of an insurance contract. However, if it is computed on the last birthday, a pro rata allowance is sometimes made for each month that has elapsed since the last birthday. In simplifying the computation in this and the next chapters, unless otherwise specified (for example, ANB is specified in Table 14 in the Appendix), it is assumed that all annuity contracts and insurance policies are made on the purchaser’s birthday. Thus, the expression “a person aged 25” means that the person reaches his 25th birthday on the purchase date.

The computation of life annuity and life insurance net premiums is primarily based on a mortality table. A mortality table is a statistical table showing the death rate of people of every age group. Although nobody can predict how long a certain individual will live or when that individual will die, studies of the mortality of people based on years of experience enable insurance companies to make a reasonably accurate prediction of the death rate of any particular age group. A large representative group of people, usually policyholders, is included in each study. The number of people living and dying in each age group is recorded and the findings are tabulated. For example, the American Experience Table was calculated from the mortality experience of the Mutual Life Insurance Company of New York. As a result of the development of actuarial science, medical discoveries, and a higher standard of living that prolongs the life of individuals, new mortality tables have become necessary and have been constructed to meet the current needs of the life insurance business. The best-known mortality tables used in recent years by life insurance companies are as follows:

American Experience Table of Mortality-First published as a part of New York law in 1868. Covered experience, 1843-58.

American Men Ultimate Mortality Table-Published in 1918. Covered experience, 1900-15.

The 1937 Standard Annuity Mortality Table-Published in 1938. Based primarily on experience, 1932-36.

Commissioners' 1941 Standard Ordinary Table of Mortality-Based on experience, 1930-40.

Commissioners' 1958 Standard Ordinary Table of Mortality-Based on experience, 1950-54.

Commissioners' 1980 Standard Ordinary Table of Mortality-Based on experience, 1970-75.

The 1983 Individual Annuity Table-Based on experience, 1971-76.

United States Total Population Mortality Table-Based on experience, 1979-81.

The basic principle involved in the computation of life annuity and life insurance problems is not affected by the use of a particular mortality table. This text will use two mortality tables, Table 12 and Table 14 in the Appendix, for illustration and updating purposes:

1. Table 12-The Commissioners' 1958 Standard Ordinary Table of Mortality, commonly referred to as the 1958 CSO Table. The Table of

Commutation Columns (Table 13 in the Appendix), interest at 4%, computed by the Society of Actuaries based on the 1958 CSO Table is used to compute various life annuity and life insurance problems in this text. For many years, in assisting the computation of the complicated and laborious life annuity and life insurance problems, commutation columns, based on the 1941 and 1958 CSO Tables, respectively, were computed and compiled by the Society of Actuaries. Table 13 is the latest available Table of Commutation Columns and is therefore used here for illustrating the basic principle of the computation.

2. Table 14-The Commissioners' 1980 Standard Ordinary Table of Mortality, or 1980 CSO Table. There was no Table of Commutation Columns, based on the 1980 CSO Table, developed by the Society of Actuaries. The Society felt that this table was not necessary, given the computing power that was available.

Table 14 is now used by most of the life insurance companies to compute premiums and benefits. For our practical reasons, the work of using computer programs to solve every problem in this text is beyond the scope of this book. Thus, only selected problems are used in illustrations.

However, the Committee On Specifications for 1980 CSO Tables, formed by the Society of Actuaries in 1981, made a slight modification of traditional commutation functions. The specifications start from a radix (for age nearest birthday functions) of 200 lives at age 99 instead of starting from a radix at age 0. (See Table 14.) Also the basic commutation functions are stated in terms of (I + i) instead of v. (See the note in Example 1, Section 19.3.)

The Committee had attempted to follow the 1958 CSO specifications as much as possible with the assumption that anyone who had developed computer programs to follow those specifications would wish to make the minimum modifications to those programs.2 This assumption is also made in our text presentation below.

In conclusion, we thus use the 1958 CSO Table and its related Table of Commutation Columns to illustrate the basic methods of computing the life annuity and life insurance problems in this chapter and the next. However, the 1980 CSO Table is also used in some illustrations to give the updated information.

Note that most of the mathematical symbols used in this chapter and the next are based on the statement of the "International Actuarial Notation," established by the Actuarial Society of America. From the 1958 CSO

Table, the symbols that are used frequently in this chapter are:

lx, the number of people living at age x, and dx, the number of people who will die between the ages x and $x + 1$[3]. The table is based on a study of 10 million people starting at the age of 0, or $l_0 = 10{,}000{,}000$. A total of 70,800 persons from the group will die between the ages of 0 and 1. Thus, the number of people living at age 1 is:

$l_1 = l_0 - d_0 = 10{,}000{,}000 - 70{,}800 = 9{,}929{,}000.$ Likewise,
$l_2 = l_1 - d_1 = 9{,}929{,}200 - 17{,}475 = 9{,}911{,}725,$ and so on. (in Table 12)

The approach of constructing the 1980 CSO Table was modified. That table is based on a group of 200 lives at age 99, or $l_{99} = 200$. The given death rate per 1,000 at age 98 is 657.98 for males. Thus,

$l_{99} = l_{98} - d_{98} = l_{98} - l_{98}(657.98/1{,}000) = l_{98}(.34202) = 200.$
$l_{98} = 200/.34202 = 585.$ Likewise,
$l_{98} = l_{97} - d_{97} = l_{97} - l_{97}(480.20/1{,}000) = l_{97}(.51980) = 585.$
$l_{97} = 585/.51980 = 1{,}125,$ and so on. (in Table 14)

EXAMPLE 1 Find the death rate for a person aged 30 (written as q_{30}) from (a) the 1958 CSO Table, and (b) the 1980 CSO Table. Which one of the death rates is the highest?

(a) From the 1958 CSO Table:

$$q_{30} = \frac{d_{30}}{l_{30}} = \frac{20{,}193}{9{,}480{,}358} = .00213 \text{ (per person)}$$

The value of q_{30} can be obtained directly from the deaths per 1,000 column. The table shows that the death rate per 1,000 at age 30 is 2.13.

$$q_{30} = \frac{2.13}{1{,}000} = .00213$$

(b) From the 1980 CSO Table:

For males, the death rate per 1,000 at age 30 is 1.73.

$q_{30} = 1.73/1{,}000 = 0.00173$

For females, the death rate per 1,000 at age 30 is 1.35.

$q_{30} = 1.35/1{,}000 = 0.00135$

The death rate computed from the 1958 CSO Table is the highest, at 0.00213 per person.

EXERCISE 19-1 REFERENCE: SECTION 19.2

A. Find the number of people in each of the following expressions (use Table 12):

1. (a) l_3 (b) l_{20} (c) l_{32} (d) l_{46} (e) l_{78} (f) l_{99}
2. (a) l_{12} (b) l_{28} (c) l_{41} (d) l_{52} (e) l_{60} (f) l_{88}
3. (a) d_3 (b) d_{20} (c) d_{32} (d) d_{47} (e) d_{68} (f) d_{99}
4. (a) d_{12} (b) d_{28} (c) d_{41} (d) d_{56} (e) d_{72} (f) d_{83}

B. Statement problems (use Table 12):

5. Find the death rate per thousand for persons aged (a) 8, (b) 20, (c) 56, (d) 70.
6. Find the death rate per thousand for persons aged (a) 10, (b) 25, (c) 61, (d) 86.
7. From a group of 50,000 now age 18, how many will probably (a) be alive at age 50, and (b) die after reaching age 50 but before reaching age 51?
8. From a group of 80,000 now age 25, how many are predicted (a) to be alive at age 65, and (b) to die after reaching age 65 but before reaching age 66?

company may be thought of as an agent that collects premiums on the purchase date and pays benefits during later years. The total cost to an insurance company on the purchase date should be divided by the number of annuitants living at that time. Thus, the net single premium for each annuitant is:

$$\frac{121{,}228.15}{97{,}165} = \$1.24765, \quad \text{or} \quad \$1.25$$

In general,

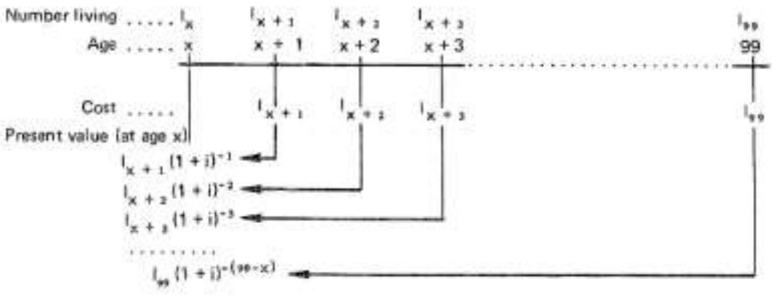
let a_x = the net single premium, or the present value, of an ordinary whole life annuity of \$1 payable at the end of each year for a person whose age now is x years

Then

$$a_x = \frac{N_{x+1}}{D_x} \qquad \textbf{(19-2)}^6$$

[6]*Proof—Formula (19-2):*

The steps necessary in finding the value of a_x are the same as those used in Example 1 and are shown symbolically as follows:



Total present value of the costs $= l_{x+1}(1+i)^{-1} + l_{x+2}(1+i)^{-2} + l_{x+3}(1+i)^{-3} + \cdots + l_{99}(1+i)^{-(99-x)}$

Let $v = (1+i)^{-1}$

The present value $= vl_{x+1} + v^2 l_{x+2} + v^3 l_{x+3} + \cdots + v^{99-x} l_{99}$

Let a_x = the net single premium, or the present value of an ordinary whole life annuity of \$1 payable at the end of each year for a person whose age now is x years.

$$a_x = \frac{\text{Total present value of the costs}}{\text{Number of annuitants at age } x \text{ on the purchase date}}$$

$$= \frac{vl_{x+1} + v^2 l_{x+2} + v^3 l_{x+3} + \cdots + v^{99-x} l_{99}}{l_x}$$

Multiply both the numerator and the denominator by v^x. Then

$$a_x = \frac{v^{x+1} l_{x+1} + v^{x+2} l_{x+2} + v^{x+3} l_{x+3} + \cdots + v^{99} l_{99}}{v^x l_x}$$

Let R = the annual payment to the annuitant
A = the net single premium for an ordinary whole life annuity that pays R per year for life

Then

$$A = Ra_x = R \cdot \frac{N_{x+1}}{D_x} \qquad \textbf{(19-3)}$$

The values of N_x and D_x, based on an interest rate of $2\frac{1}{2}\%$ compounded annually, are listed in Table 13.

EXAMPLE 2 Refer to Example 1. Solve using Formula (19-3).

$R = \$1, x = 95$

$$A = 1 \cdot \frac{N_{95+1}}{D_{95}} = \frac{N_{96}}{D_{95}} = \frac{11{,}610.1087}{9{,}305.5630} = 1.24765, \quad \text{or} \quad \$1.25$$

NOTE: According to the modifications made by the Committee on Specifications for the 1980 CSO Tables, the division N_{96}/D_{95} may also be performed by using computers in a different manner. See Section 19.6, Example 1.

EXAMPLE 3 A person aged 35 wishes to purchase an ordinary whole life annuity that will pay \$1,000 at the age of 36 and the same amount at the end of each year thereafter for life. Find the net single premium of the annuity.

Since the first payment is to be made one year after the date of purchase, the contract is an ordinary whole life annuity. The interest rate is assumed to be $2\frac{1}{2}\%$.

$x = 35, R = \$1{,}000$

Substituting the above values in Formula (19-3),

$$A = R \cdot \frac{N_{x+1}}{D_x} = R \cdot \frac{N_{35+1}}{D_{35}} = 1{,}000 \cdot \frac{N_{36}}{D_{35}}$$

$$= 1{,}000 \cdot \frac{89{,}956{,}987.56}{3{,}949{,}851.09} = \$22{,}774.78 \quad \text{(Table 13)}$$

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[5]*Proof—(continued):*

According to Section 19.3 (footnote 4), $D_x = v^x l_x$. Thus,

$$a_x = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \cdots + D_{99}}{D_x}$$

Let $N_x = D_x + D_{x+1} + D_{x+2} + \cdots + D_{99}$. Then
$N_{x+1} = D_{x+1} + D_{x+2} + \cdots + D_{99}$ and

$$a_x = \frac{N_{x+1}}{D_x} \qquad (19\text{-}2)$$

Note: If we multiply both numerator and denominator of a_x by $(1+i)^{100-x}$ instead of v^x, we then have:

$D_x = (1+i)^{100-x} \cdot l_x$, $D_{x+1} = (1+i)^{100-(x+1)} \cdot l_{x+1}$

and so on. Also, see footnote 5.

EXAMPLE 4 A person aged 25 has \$50,000. If the money is used to purchase an ordinary whole life annuity with the first payment payable one year from the purchase date, what is the size of each annual payment?

The net single premium of the annuity is known, \$50,000 = A, $x = 25$, $R = ?$

Substituting the above values in Formula (19-3),

$$50{,}000 = R \cdot \frac{N_{25+1}}{D_{25}} = R \cdot \frac{N_{26}}{D_{25}}$$

$$R = 50{,}000 \cdot \frac{D_{25}}{N_{26}} = 50{,}000 \cdot \frac{5{,}165{,}007.95}{134{,}674{,}488.96} = \$1{,}917.59 \text{ (per year)}$$

EXERCISE 19-3 REFERENCE: SECTION 19.4A

1. A person aged 20 wishes to purchase an ordinary whole life annuity that will pay \$3,000 at age 21 and the same amount at the end of each year thereafter. What is the net single premium of the annuity?
2. A person aged 32 wants to buy an annuity that will pay \$2,500 annually for life with the first payment to be made one year after the purchase date. What is the purchase price (the net single premium)?
3. Find the net single premium for an ordinary whole life annuity of \$2,500 per year for a person now aged 60.
4. What is the net single premium for an ordinary whole life annuity of \$1,000 per year for a person now aged 30?
5. Find the net single premium for an ordinary whole life annuity of \$2,000 payable at the end of each year for a person now aged 45.
6. Compute the net single premium for an ordinary whole life annuity of \$3,000 per year if the annuity is bought by a person now aged 40.
7. What is the net single premium for an ordinary whole life annuity of \$4,000 per year for a person now aged 50?
8. Find the net single premium for an ordinary whole life annuity of \$5,000 payable at the end of each year for a person now aged 55.
9. If the net single premium of an ordinary whole life annuity for a person now aged 30 is \$20,000, what is the size of each annual payment?
10. Refer to Problem 9. What is the size of each annual payment if the annuity is purchased by a person aged 22?
11. A person aged 27 has \$120,000. If the money is used to buy an ordinary whole life annuity with the first payment to be made one year from the purchase date, what is the size of each annual payment?
12. A person aged 48 paid \$70,000 for an ordinary whole life annuity with the first payment to be made in one year. Find the size of the annual payment.

B. Whole Life Annuity Due

The first payment of an ordinary whole life annuity is made one year after the date of purchase, whereas the first payment of a whole life annuity *due* is made at the *time of* *purchase.* Thus, if the annual payment is \$1, the net single premium of a whole life annuity due on the purchase date is *\$1 more* than the net single premium of an ordinary whole life annuity.

Let $\ddot{a}_x$ = the net single premium, or the present value, of a whole life annuity due of \$1 payable now and each year hereafter for life for a person now aged x years

Then

$$\ddot{a}_x = 1 + a_x, \quad \text{or}$$

$$\ddot{a} = \frac{N_x}{D_x} \qquad (19\text{-}4)^7$$

Let $A(\text{due})$ = the net single premium or the present value of a whole life annuity due that will pay R per year for life

$$A(\text{due}) = R\ddot{a}_x = R \cdot \frac{N_x}{D_x} \qquad (19\text{-}5)$$

EXAMPLE 5 A person aged 35 wishes to purchase a whole life annuity that will pay \$1,000 now and the same amount at the end of each year thereafter for life. Find the net single premium of the annuity.

$x = 35$, $R = \$1{,}000$. Since the first payment is made now, this is a whole life annuity due problem. The net single premium is:

$$A(\text{due}) = R\ddot{a}_x = 1{,}000 \cdot \frac{N_{35}}{D_{35}} = 1{,}000 \cdot \frac{93{,}906{,}838.64}{3{,}949{,}851.09}$$

$$= \$23{,}774.78$$

NOTE: Example 5 is identical to Example 3, except that the first payment in Example 5 is made at the time of purchase. Since the annuitant receives \$1,000 at the time of purchase, the cost is also \$1,000 higher than the cost in Example 3. Thus, the difference between the two purchase prices in Examples 5 and 3 is \$1,000, or \$23,774.78 − \$22,774.78.

EXAMPLE 6 A person aged 25 owes a life insurance company \$50,000. If the company allows that person to discharge the obligation by annual payments payable for life, with the first payment due now, what is the size of the annual payment?

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[7] *Proof—Formula (19-4):*

$$\ddot{a}_x = 1 + a_x = 1 + \frac{N_{x+1}}{D_x} = \frac{D_x + N_{x+1}}{D_x}$$

$$= \frac{D_x + (D_{x+1} + D_{x+2} + \cdots + D_{99})}{D_x} = \frac{N_x}{D_x} \qquad (19\text{-}4)$$

The above result may also be obtained by a method similar to that used in obtaining the ordinary whole life annuity [Formula (19-2)].

The present value of the annuity $A(\text{due})$ is known.

$A(\text{due}) = \$50{,}000$, $x = 25$, $R = ?$ (annual payment)

Substituting the values in Formula (19-5),

$$A(\text{due}) = R\ddot{a}_x = R \cdot \frac{N_x}{D_x} = R \cdot \frac{N_{25}}{D_{25}}$$

Thus,

$$50{,}000 = R \cdot \frac{N_{25}}{D_{25}}$$

$$R = 50{,}000 \cdot \frac{D_{25}}{N_{25}} = 50{,}000 \cdot \frac{5{,}165{,}007.95}{139{,}839{,}496.91} = \$1{,}846.76$$

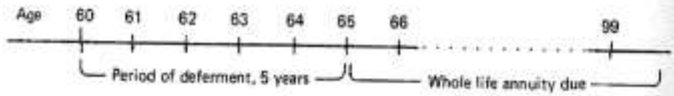
NOTE: The concept of finding the annual payments of a whole life annuity due is important in finding the *annual* premiums payable for life for a life insurance policy. (See Section 20.2B.)

★C. Deferred Whole Life Annuity

The first payment of a deferred whole life annuity begins after a period of more than one year has elapsed from the date of purchase. The period of deferment may be expressed in two ways:

1. It is the period from the date of purchase to the date of the first payment. The payments thus form a whole life annuity *due* with the term beginning on the date of the first payment.
2. It is the period from the date of purchase to the date that is one year prior to the date of the first payment. The payments thus form an *ordinary* whole life annuity.

For convenience in this chapter, the period of deferment is expressed from the date of purchase to the date of the first payment (item 1 above). For example, if a person aged 60 buys a whole life annuity with the first payment to be made at 65 years of age, the period of deferment is considered to be five years. The annuity is a deferred whole life annuity *due*, with the term beginning at age 65.



Let k = the period of deferment in years

${}_{k|}\ddot{a}_x$ = the net single premium, or the present value, of a deferred annuity of \$1 per year for life, with the first payment at the end of the deferment period, k years, for a person now aged x years

Then

$$ {}_{k|}\ddot{a}_x = \frac{N_{x+k}}{D_x} \qquad \textbf{(19-6)}[8] $$

Let A(defer.) = the net single premium, or the present value, of a whole life annuity that will pay R per year after k years

Then

$$ A(\textbf{defer.}) = R \cdot {}_{k|}\ddot{a}_x = R \cdot \frac{N_{x+k}}{D_x} \qquad \textbf{(19-7)} $$

EXAMPLE 7 A person aged 25 wishes to purchase a whole life annuity that will pay \$3,000 a year for life. The first payment is due at age 65. Find the net single premium of the annuity.

The period of deferment $k = 65 - 25 = 40$ (years), $x = 25$, $R = \$3{,}000$

Substituting the values in Formula (19-7),

$$ A(\text{defer.}) = 3{,}000 \cdot \frac{N_{25+40}}{D_{25}} = 3{,}000 \cdot \frac{N_{65}}{D_{25}} $$

$$ = 3{,}000 \cdot \frac{15{,}077{,}832.60}{5{,}165{,}007.95} = \$8{,}757.68 $$

EXAMPLE 8 A 14-year-old boy inherited \$50,000. If he uses the money to purchase a whole life annuity with the first payment due at age 25, what will be the size of each payment?

A(defer.) = \$50,000, $x = 14$, $k = 25 - 14 = 11$ (years),
$x + k = 14 + 11 = 25$, $R = ?$ (per year)

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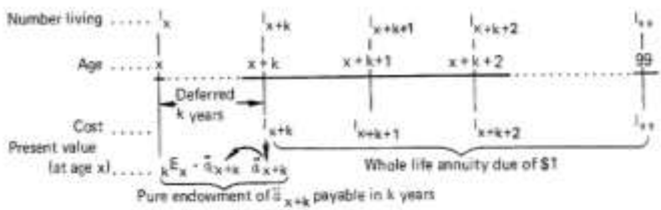
[8] *Proof—Formula (19-6):*

The value of ${}_{k|}\ddot{a}_x$ is obtained as follows:



The following result is obtained from the above diagram:

$$ {}_{k|}\ddot{a}_x = {}_kE_x \cdot \ddot{a}_{x+k} = \frac{D_{x+k}}{D_x} \cdot \frac{N_{x+k}}{D_{x+k}} = \frac{N_{x+k}}{D_x} \qquad \text{(19-6) [see Formulas (19-1) and (19-4)]} $$

Note: Formula (19-6) may also be proved by using a method similar to that used in proving Formula (19-2).

Substituting the values in Formula (19-7),

$$50{,}000 = R \cdot \frac{N_{25}}{D_{14}}$$

$$R = 50{,}000 \cdot \frac{D_{14}}{N_{25}} = 50{,}000 \cdot \frac{6{,}905{,}108.16}{139{,}839{,}496.91} = \$2{,}468.94$$

EXERCISE 19-4 REFERENCE: SECTION 19.4B AND C

1. Refer to Problem 1 of Exercise 19-3. What is the net single premium if the annuity pays $3,000 annually starting at age 20?
2. Refer to Problem 2 of Exercise 19-3. If the first payment is made on the purchase date, what is the purchase price?
3. A person aged 45 wishes to buy a whole life annuity that will pay $1,000 now and the same amount at the end of each year thereafter for life. What is the net single premium of the annuity?
4. Find the net single premium of a whole life annuity of $2,000 per year for someone aged 30. Assume that the first $2,000 is payable to the annuitant on the purchase date.
5. What is the net single premium of a whole life annuity due of $3,000 per year for a person aged 18?
6. What is the net single premium of a whole life annuity due of $4,000 per year for a person aged 70?
7. If the net single premium of a whole life annuity due is $20,000, what is the size of the annual payment for a person now aged 30?
8. Refer to Problem 7. If the person is 22 years old now, what is the size of the annual payment?
9. A person, aged 55, purchases a life insurance policy. The net single premium (purchase price) of the policy is $2,000. If the insurance company allows the premium to be paid by equal annual payments for life, with the first payment due now (on the purchase date), find the size of the annual payment.
10. A person, who is the beneficiary of a $10,000 insurance policy, decides to use the money to purchase a whole life annuity with the first payment due now. If the person is 26 years old, what will be the size of each annual payment?

★11. Find the net single premium of a whole life annuity of $1,000 per year for a person now aged 36 if the first payment is to be made 10 years from now.

★12. A person aged 42 wishes to buy a whole life annuity of $500 payable at the beginning of each year. The first payment is due at age 55. What is the net single premium of the annuity?

★13. Refer to Problem 3. What is the net single premium of the annuity if the first payment is to be made at age 50?

★14. Refer to Problem 4. What is the net single premium of the annuity if the first payment is to be made at age 38?

★15. Refer to Problem 10. If the first payment is to be made at age 40, what will be the size of each annual payment?

★16. A young man aged 20 has $15,000. If he wishes to use the money to buy a whole life annuity with the first payment to be made at age 35, what is the size of each payment?

19.5 TEMPORARY LIFE ANNUITIES

When the payments of a life annuity cease at the end of a certain number of years, even though the annuitant is still living, the annuity is called a *temporary life annuity*. Like whole life annuities, temporary life annuities may be classified as ordinary, due, and deferred, depending upon the date of the first payment.

A. Ordinary Temporary Life Annuity (Immediate Temporary Life Annuity)

The first annual payment of an *ordinary temporary life annuity* is made one year after the date of purchase. Thus, if a person now aged x purchases an ordinary temporary life annuity, the first annual payment will be made to him at $x + 1$ years of age.

Let n = the number of payments
$a_{x:\overline{n}|}$ = the net single premium, or present value, at age x of an ordinary temporary life annuity of \$1 payable each year for n annual payments

Then

$$a_{x:\overline{n}|} = \frac{N_{x+1} - N_{x+n+1}}{D_x} \qquad \textbf{(19-8)}^{9}$$

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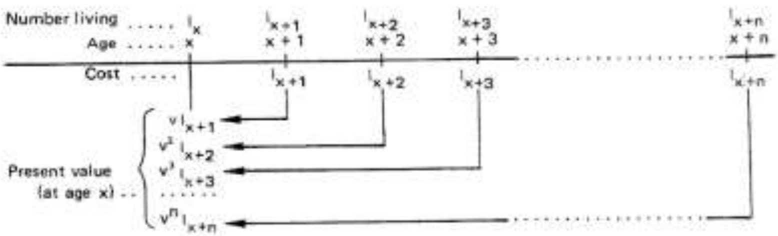
[9] *Proof—Formula (19-8):*

The value of $a_{x:\overline{n}|}$ is obtained as follows:



The net single premium to each annuitant at the present time (age x) is:

$$a_{x:\overline{n}|} = \frac{vl_{x+1} + v^2l_{x+2} + v^3l_{x+3} + \cdots + v^nl_{x+n}}{l_x}$$

Multiply both the numerator and the denominator by v^x, and substitute commutation symbols:

$$a_{x:\overline{n}|} = \frac{D_{x+1} + D_{x+2} + D_{x+3} + \cdots + D_{x+n}}{D_x}$$

Since $N_{x+1} = D_{x+1} + D_{x+2} + D_{x+3} + \cdots + D_{x+n} + D_{x+n+1} + \cdots + D_{99}$, and
$N_{x+n+1} = D_{x+n+1} + \cdots + D_{99}$,

the difference between N_{x+1} and N_{x+n+1} equals the numerator in the fraction on the right side in the above equation. Thus, the equation may be written in the following simple manner:

$$a_{x:\overline{n}|} = \frac{N_{x+1} - N_{x+n+1}}{D_x} \qquad (19\text{-}8)$$

Let A(tem.) = net single premium, or present value, of an ordinary temporary life annuity that will pay R per year

Then

$$A(\text{tem.}) = Ra_{x:\overline{n}|} = R \cdot \frac{N_{x+1} - N_{x+n+1}}{D_x} \tag{19-9}$$

EXAMPLE 1 What is the net single premium of a five-year ordinary temporary life annuity of \$2,000 per year for a person aged 20 if the first payment is to be made at age 21?

$x = 20$, $n = 5$, $R = \$2{,}000$

$$A(\text{tem.}) = Ra_{x:\overline{n}|} = 2{,}000 \cdot \frac{N_{20+1} - N_{20+5+1}}{D_{20}} = 2{,}000 \cdot \frac{N_{21} - N_{26}}{D_{20}}$$

$$= 2{,}000 \cdot \frac{161{,}928{,}780.91 - 134{,}674{,}488.96}{5{,}898{,}264.97}$$

$$= 2{,}000 \cdot \frac{27{,}254{,}291.95}{5{,}898{,}264.97} = \$9{,}241.46$$

B. Temporary Life Annuity Due

The first payment of a *temporary life annuity due* is made on the date of purchase. If the periodic payment is \$1, the present value of the first payment is also \$1. The remaining payments form an ordinary temporary life annuity. Let n = the number of payments. The number of the remaining payments is $n - 1$, and the present value of the remaining payments of \$1 each is $a_{x:\overline{n-1}|}$.

Let $\ddot{a}_{x:\overline{n}|}$ = the net single premium, or the present value, at age x, of a temporary life annuity *due* of \$1 payable each year for n annual payments

Then

$$\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}, \quad \text{or}$$

$$\ddot{a}_{x:\overline{n}|} = \frac{N_x - N_{x+n}}{D_x} \tag{19-10}$$ [10]

The net single premium of a temporary life annuity due of R per year for n payments is:

$$A(\text{tem. due}) = R\ddot{a}_{x:\overline{n}|} = R \cdot \frac{N_x - N_{x+n}}{D_x} \tag{19-11}$$

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[10] *Proof—Formula (19-10):*

$$\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|} = 1 + \frac{N_{x+1} - N_{x+(n-1)+1}}{D_x} = \frac{D_x + N_{x+1} - N_{x+n}}{D_x}$$

Since $D_x + N_{x+1} = N_x$ (see Section 19.4B),

then $\ddot{a}_{x:\overline{n}|} = \dfrac{N_x - N_{x+n}}{D_x}$ (Formula 19-10)

EXAMPLE 2 What is the net single premium for a six-year temporary life annuity of \$2,000 per year for a person aged 20 if the first payment is due now?

$x = 20, n = 6, R = \$2{,}000$

$$A(\text{tem. due}) = 2{,}000 \cdot \frac{N_{20} - N_{20+6}}{D_{20}} = 2{,}000 \cdot \frac{N_{20} - N_{26}}{D_{20}}$$

$$= 2{,}000 \cdot \frac{167{,}827{,}045.88 - 134{,}674{,}488.96}{5{,}898{,}264.97}$$

$$= \frac{2{,}000(33{,}152{,}556.92)}{5{,}898{,}264.97} = \$11{,}241.46$$

NOTE: The difference between the answers in Example 1 and Example 2 is \$2,000, which is the value of the first payment on the date of purchase.

EXAMPLE 3 The purchase price (or the net single premium) of a life insurance policy issued to a person aged 25 is \$3,396.49. That person will pay the premium by making equal annual payments for 20 years or for life, whichever is the shorter of the two periods. If the first payment is due now (the purchase date), what is the size of the annual payment?

$A(\text{tem. due}) = \$3{,}396.49, x = 25, n = 20, R = ?$

Substituting the above values in Formula (19-11),

$$3{,}396.49 = R \cdot \frac{N_{25} - N_{25+20}}{D_{25}}$$

$$R = 3{,}396.49 \cdot \frac{D_{25}}{N_{25} - N_{45}}$$

$$= 3{,}396.49 \cdot \frac{5{,}165{,}007.95}{139{,}839{,}496.91 - 58{,}927{,}803.08}$$

$$= \$216.82 \text{ (annual payment)}$$

NOTE: The concept of finding the annual payments of a temporary life annuity due is important in finding the annual premiums payable for a limited number of payments for a life insurance policy. (See Example 3, Section 20.2B.)

★C. Deferred Temporary Life Annuity

The first annual payment of a *deferred temporary life annuity* is made after a period of k (more than 1) years, or at age $x + k$ of the annuitant, if then still living. The n annual payments form a temporary life annuity *due*.

Let ${}_{k|}\ddot{a}_{x:\overline{n}|}$ = the net single premium, or the present value, at age x, of a deferred temporary life annuity due of \$1 per year for n annual payments with the first payment at the end of k years, or at age $x + k$

Then

$$_{k|}\ddot{a}_{x:\overline{n}|} = \frac{N_{x+k} - N_{x+k+n}}{D_x} \qquad (19\text{-}12)^{11}$$

The net single premium of a deferred temporary life annuity of R per year is:

$$A(\textbf{tem. defer.}) = R \cdot \frac{N_{x+k} - N_{x+k+n}}{D_x} \qquad (19\text{-}13)$$

EXAMPLE 4 What is the net single premium for an eight-year temporary life annuity of $1,000 per year for a person aged 25 if the first payment is due at age 45?

$x = 25, k = 45 - 25 = 20, n = 8, R = \$1{,}000$

$$A(\text{tem. defer.}) = 1{,}000 \cdot \frac{N_{25+20} - N_{25+20+8}}{D_{25}} = 1{,}000 \cdot \frac{N_{45} - N_{53}}{D_{25}}$$

$$= 1{,}000 \cdot \frac{58{,}927{,}803.08 - 37{,}504{,}037.97}{5{,}165{,}007.95}$$

$$= \$4{,}147.87$$

EXERCISE 19-5 REFERENCE: SECTION 19.5

1. What is the net single premium of a 10-year ordinary temporary life annuity of $5,000 per year for a person aged 30 if the first payment is to be made at age 31?
2. Find the net single premium of an ordinary temporary life annuity of 15 payments of $1,500 each for a person aged 45 if the first payment is to be made one year after the purchase date.
3. How much would a person aged 40 have to pay for a 20-year ordinary temporary life annuity of $2,000 each year if the first payment is to be made at age 41?
4. What is the net single premium of a five-year ordinary temporary life annuity of $1,200 per year for a person aged 24 if the first payment is to be made at age 25?
5. A person aged 42 has $8,000. If the money is used to buy a 10-year ordinary temporary life annuity, what is the size of the annual payment to that individual?
6. A person aged 20 owes a life insurance company $1,500. The company allows the debt to be paid by 18 equal annual payments or for life, whichever period is the shorter. If the first payment is due at age 21, what is the size of the annual payment?

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[11] *Proof—Formula (19-12):*

The value of $_{k|}\ddot{a}_{x:\overline{n}|}$ may be obtained by a method similar to the proof for Formula (19-6), in Section 19.4C, as follows:

$$_{k|}\ddot{a}_{x:\overline{n}|} = {}_kE_x \cdot \ddot{a}_{x+k:\overline{n}|} = \frac{D_{x+k}}{D_x} \cdot \frac{N_{x+k} - N_{x+k+n}}{D_{x+k}} = \frac{N_{x+k} - N_{x+k+n}}{D_x} \qquad (19\text{-}12)$$

Note: The value of $\ddot{a}_{x+k:\overline{n}|}$ is the net single premium of a temporary life annuity due of $1 payable for n annual payments for a person now aged $x + k$. The value of $_{k|}\ddot{a}_{x:\overline{n}|}$ is equal to the present value (age x) of the pure endowment of $\ddot{a}_{x+k:\overline{n}|}$ payable in k years. [See Formulas (19-1) and (19-10).]

7. Find the cost of an ordinary temporary life annuity of $1,000 per year for 15 payments for a person aged 40 if the first payment is due now.
8. What is the net single premium of a 10-year ordinary temporary life annuity of $3,000 per year for a person aged 35 if the first payment is made on the date of purchase?
9. Refer to Problem 1. What is the net single premium if the first payment of the life annuity is made at age 30?
10. Refer to Problem 2. What is the net single premium if the first payment of the life annuity is made at age 45?
11. Refer to Problem 5. If that person buys a 10-year temporary life annuity due, what is the size of the annual payment?
12. Refer to Problem 6. If the first of the 18 annual payments is due at age 20, find the size of the annual payment.

★13. Find the net single premium for a 20-year temporary life annuity of $500 per year for a person aged 28 if the first payment is due at age 40.

★14. What is the net single premium for a 15-year temporary life annuity of $2,000 per year for a person now aged 32 if the first payment is to be made at age 50?

★15. Refer to Problem 5. If a 10-year temporary life annuity is bought with the first payment to be made to the individual at age 50, what is the size of the annual payment?

★16. A person aged 20 paid $1,500 to buy a 12-year temporary life annuity with the first payment to be made to him at age 36. What will be the size of each annual payment?