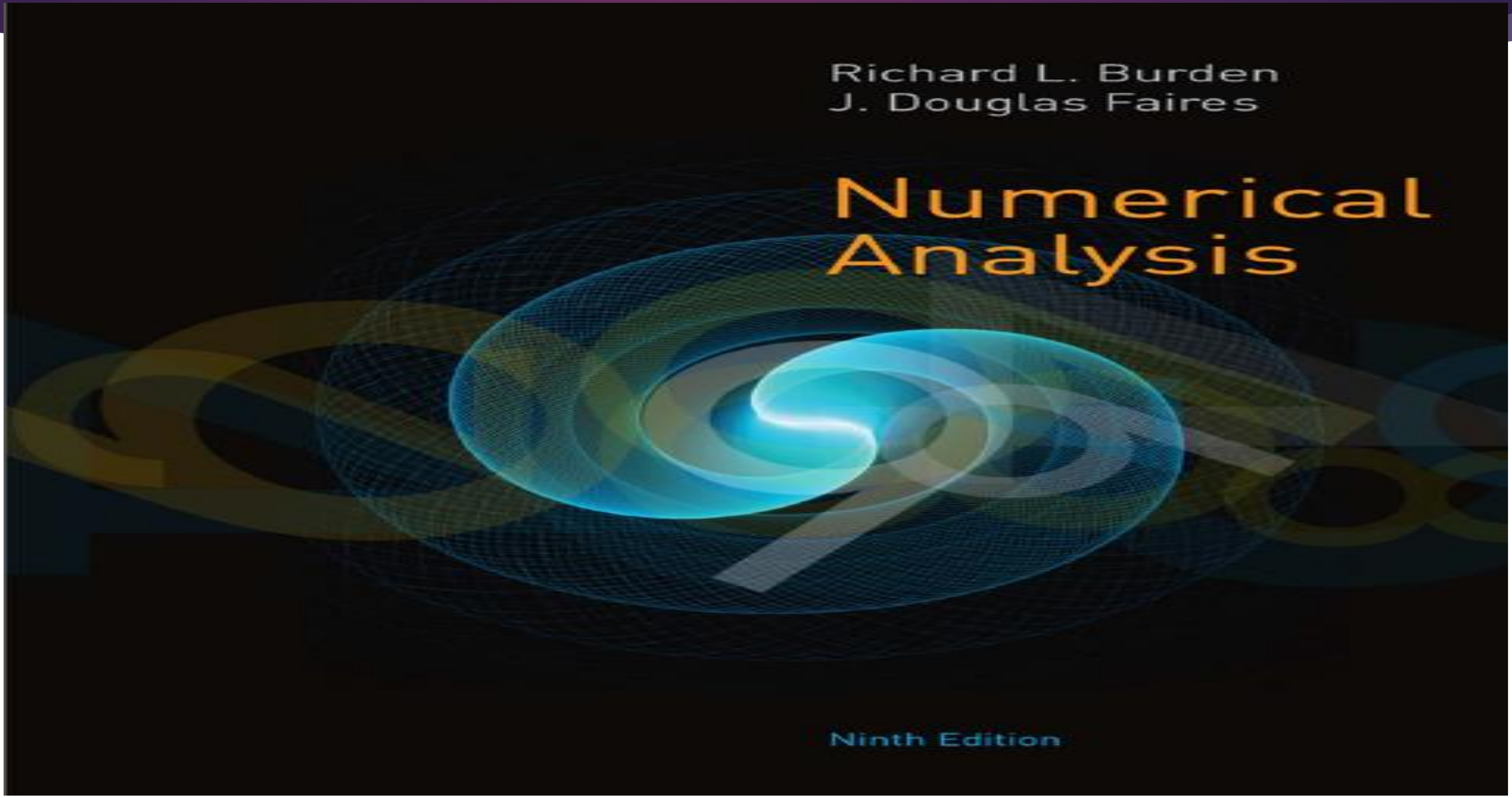
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Numerical
Analysis
Ninth Edition

# Numerical Analysis

## The Secant Method

Newton's method is an extremely powerful technique, but it has a major weakness: the need to know the value of the derivative of $f$ at each approximation. Frequently, $f'(x)$ is far more difficult and needs more arithmetic operations to calculate than $f(x)$.

To circumvent the problem of the derivative evaluation in Newton's method, we introduce a slight variation. By definition,

$$f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}.$$

If $p_{n-2}$ is close to $p_{n-1}$, then

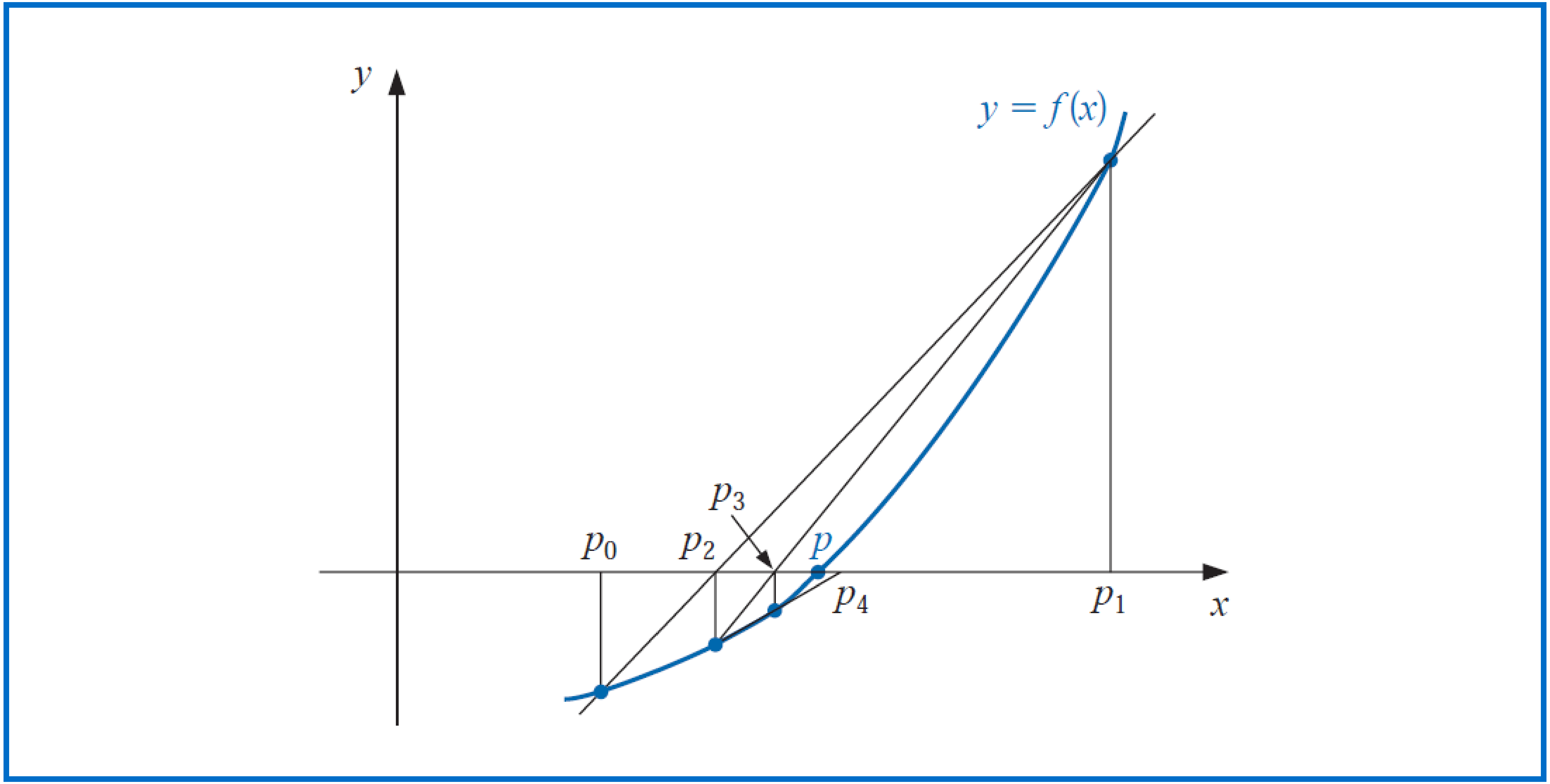
$$f'(p_{n-1}) \approx \frac{f(p_{n-2}) - f(p_{n-1})}{p_{n-2} - p_{n-1}} = \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}.$$

Using this approximation for $f'(p_{n-1})$ in Newton's formula gives

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}. \tag{2.12}$$

This technique is called the **Secant method** and is presented in Algorithm 2.4. (See Figure 2.10.) Starting with the two initial approximations $p_0$ and $p_1$, the approximation $p_2$ is the $x$-intercept of the line joining $(p_0, f(p_0))$ and $(p_1, f(p_1))$. The approximation $p_3$ is the $x$-intercept of the line joining $(p_1, f(p_1))$ and $(p_2, f(p_2))$, and so on. Note that only one function evaluation is needed per step for the Secant method after $p_2$ has been determined. In contrast, each step of Newton's method requires an evaluation of both the function and its derivative.

**Figure 2.10**



## Secant

To find a solution to $f(x) = 0$ given initial approximations $p_0$ and $p_1$:

INPUT initial approximations $p_0, p_1$; tolerance $TOL$; maximum number of iterations $N_0$.

OUTPUT approximate solution $p$ or message of failure.

*Step 1* Set $i = 2$;
$q_0 = f(p_0)$;
$q_1 = f(p_1)$.

*Step 2* While $i \le N_0$ do Steps 3–6.

*Step 3* Set $p = p_1 - q_1(p_1 - p_0)/(q_1 - q_0)$. *(Compute $p_i$.)*

*Step 4* If $|p - p_1| < TOL$ then
OUTPUT $(p)$; *(The procedure was successful.)*
STOP.

*Step 5* Set $i = i + 1$.

*Step 6* Set $p_0 = p_1$; *(Update $p_0, q_0, p_1, q_1$.)*
$q_0 = q_1$;
$p_1 = p$;
$q_1 = f(p)$.

*Step 7* OUTPUT ('The method failed after $N_0$ iterations, $N_0$ =', $N_0$);
*(The procedure was unsuccessful.)*
STOP.

**Example 2** Use the Secant method to find a solution to $x = \cos x$, and compare the approximations with those given in Example 1 which applied Newton's method.

**Table 2.5**

| $n$ | Secant $p_n$ |
|---|---|
| 0 | 0.5 |
| 1 | 0.7853981635 |
| 2 | 0.7363841388 |
| 3 | 0.7390581392 |
| 4 | 0.7390851493 |
| 5 | 0.7390851332 |

| $n$ | Newton $p_n$ |
|---|---|
| 0 | 0.7853981635 |
| 1 | 0.7395361337 |
| 2 | 0.7390851781 |
| 3 | 0.7390851332 |
| 4 | 0.7390851332 |

***Solution*** In Example 1 we compared fixed-point iteration and Newton's method starting with the initial approximation $p_0 = \pi/4$. For the Secant method we need two initial approximations. Suppose we use $p_0 = 0.5$ and $p_1 = \pi/4$. Succeeding approximations are generated by the formula

$$p_n = p_{n-1} - \frac{(p_{n-1} - p_{n-2})(\cos p_{n-1} - p_{n-1})}{(\cos p_{n-1} - p_{n-1}) - (\cos p_{n-2} - p_{n-2})}, \quad \text{for } n \geq 2.$$

These give the results in Table 2.5. ■

Comparing the results in Table 2.5 from the Secant method and Newton's method, we see that the Secant method approximation $p_5$ is accurate to the tenth decimal place, whereas Newton's method obtained this accuracy by $p_3$. For this example, the convergence of the Secant method is much faster than functional iteration but slightly slower than Newton's method. This is generally the case. (See Exercise 14 of Section 2.4.)

Newton's method or the Secant method is often used to refine an answer obtained by another technique, such as the Bisection method, since these methods require good first approximations but generally give rapid convergence.

# CHAPTER 3

# Interpolation and Polynomial Approximation

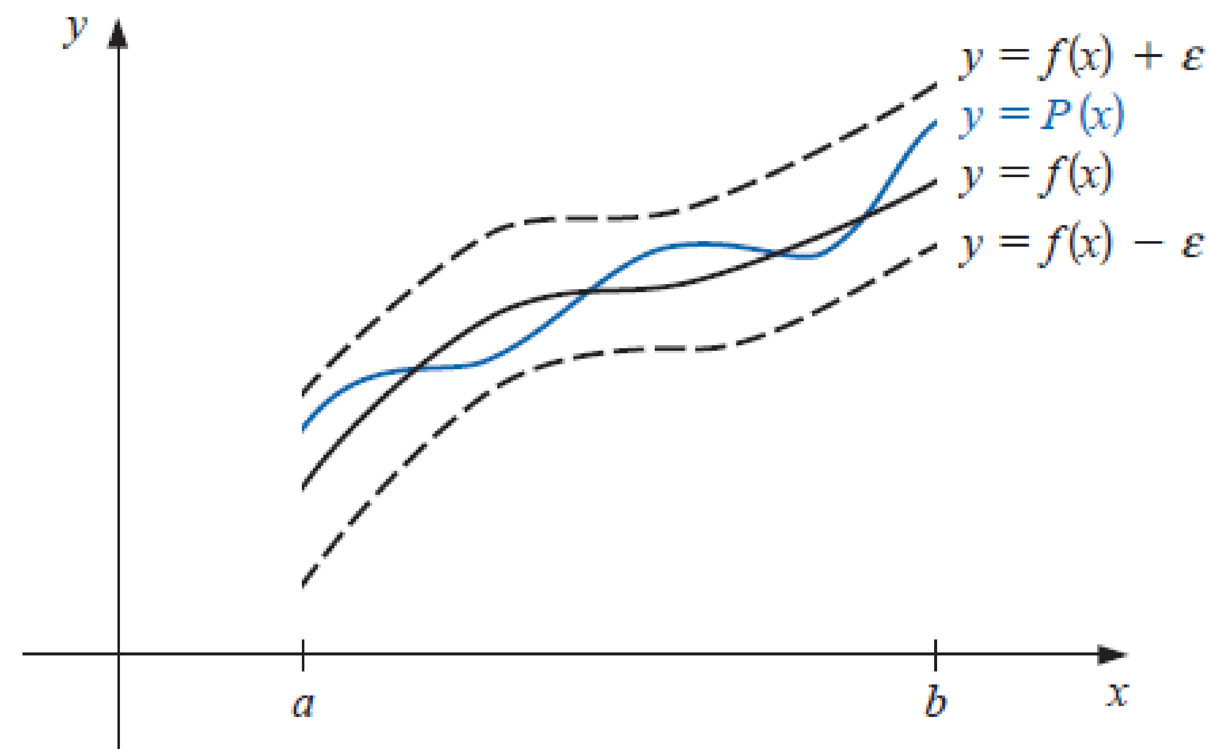
# 3.1 Interpolation and the Lagrange Polynomial

One of the most useful and well-known classes of functions mapping the set of real numbers into itself is the *algebraic polynomials*, the set of functions of the form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $n$ is a nonnegative integer and $a_0, \ldots, a_n$ are real constants. One reason for their importance is that they uniformly approximate continuous functions. By this we mean that given any function, defined and continuous on a closed and bounded interval, there exists a polynomial that is as "close" to the given function as desired. This result is expressed precisely in the Weierstrass Approximation Theorem. (See Figure 3.1.)

Figure 3.1



## Lagrange Interpolating Polynomials

The problem of determining a polynomial of degree one that passes through the distinct points $(x_0, y_0)$ and $(x_1, y_1)$ is the same as approximating a function $f$ for which $f(x_0) = y_0$ and $f(x_1) = y_1$ by means of a first-degree polynomial **interpolating**, or agreeing with, the values of $f$ at the given points. Using this polynomial for approximation within the interval given by the endpoints is called polynomial **interpolation**.

Define the functions

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} \quad \text{and} \quad L_1(x) = \frac{x - x_0}{x_1 - x_0}.$$

The linear **Lagrange interpolating polynomial** through $(x_0, y_0)$ and $(x_1, y_1)$ is

$$P(x) = L_0(x)f(x_0) + L_1(x)f(x_1) = \frac{x - x_1}{x_0 - x_1}f(x_0) + \frac{x - x_0}{x_1 - x_0}f(x_1).$$

$$P(x) = L_0(x) f(x_0) + L_1(x) f(x_1) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1).$$

Note that

$$L_0(x_0) = 1, \quad L_0(x_1) = 0, \quad L_1(x_0) = 0, \quad \text{and} \quad L_1(x_1) = 1,$$

which implies that

$$P(x_0) = 1 \cdot f(x_0) + 0 \cdot f(x_1) = f(x_0) = y_0$$

and

$$P(x_1) = 0 \cdot f(x_0) + 1 \cdot f(x_1) = f(x_1) = y_1.$$

So $P$ is the unique polynomial of degree at most one that passes through $(x_0, y_0)$ and $(x_1, y_1)$.

**Example 1** Determine the linear Lagrange interpolating polynomial that passes through the points (2, 4) and (5, 1).

***Solution*** In this case we have

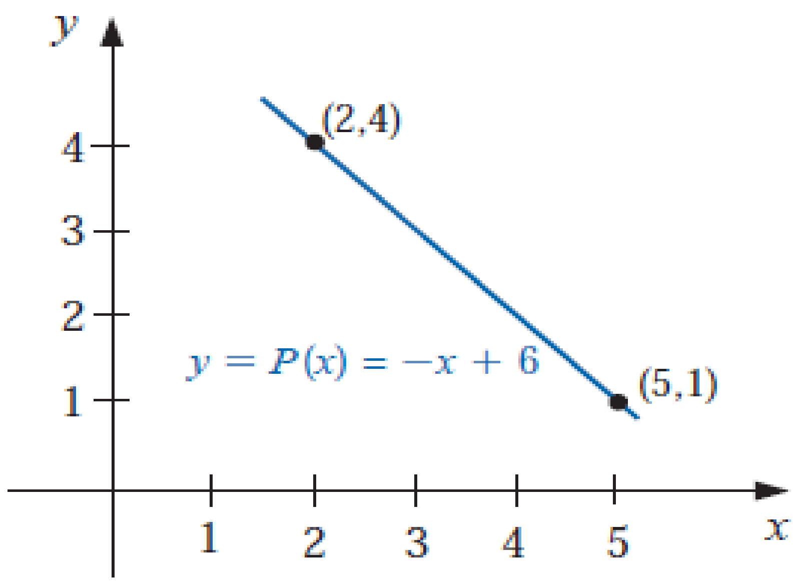
$$L_0(x) = \frac{x-5}{2-5} = -\frac{1}{3}(x-5) \quad \text{and} \quad L_1(x) = \frac{x-2}{5-2} = \frac{1}{3}(x-2),$$

so

$$P(x) = -\frac{1}{3}(x-5)\cdot 4 + \frac{1}{3}(x-2)\cdot 1 = -\frac{4}{3}x + \frac{20}{3} + \frac{1}{3}x - \frac{2}{3} = -x + 6.$$

The graph of $y = P(x)$ is shown in Figure 3.3. ■

Figure 3.3

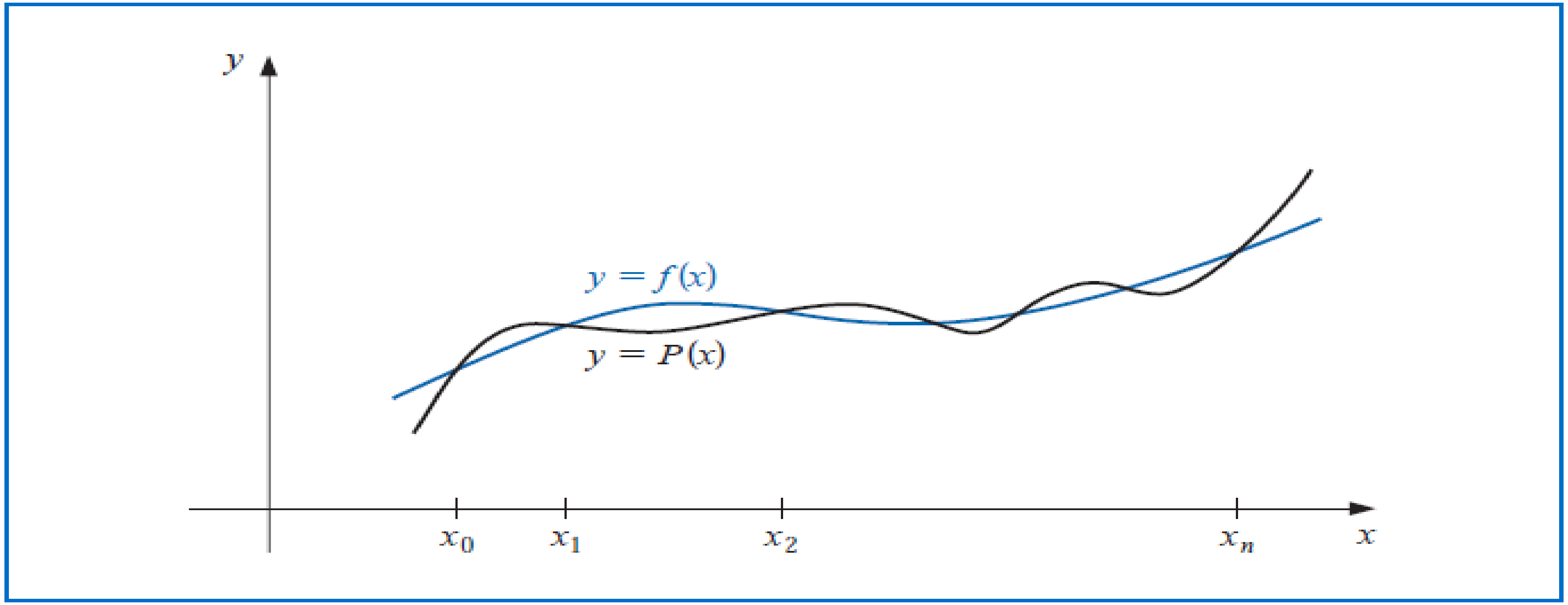


To generalize the concept of linear interpolation, consider the construction of a polynomial of degree at most $n$ that passes through the $n + 1$ points

$$(x_0, f(x_0)),\ (x_1, f(x_1)), \ldots, (x_n, f(x_n)).$$

(See Figure 3.4.)

**Figure 3.4**



In this case we first construct, for each $k = 0, 1, \ldots, n$, a function $L_{n,k}(x)$ with the property that $L_{n,k}(x_i) = 0$ when $i \neq k$ and $L_{n,k}(x_k) = 1$. To satisfy $L_{n,k}(x_i) = 0$ for each $i \neq k$ requires that the numerator of $L_{n,k}(x)$ contain the term

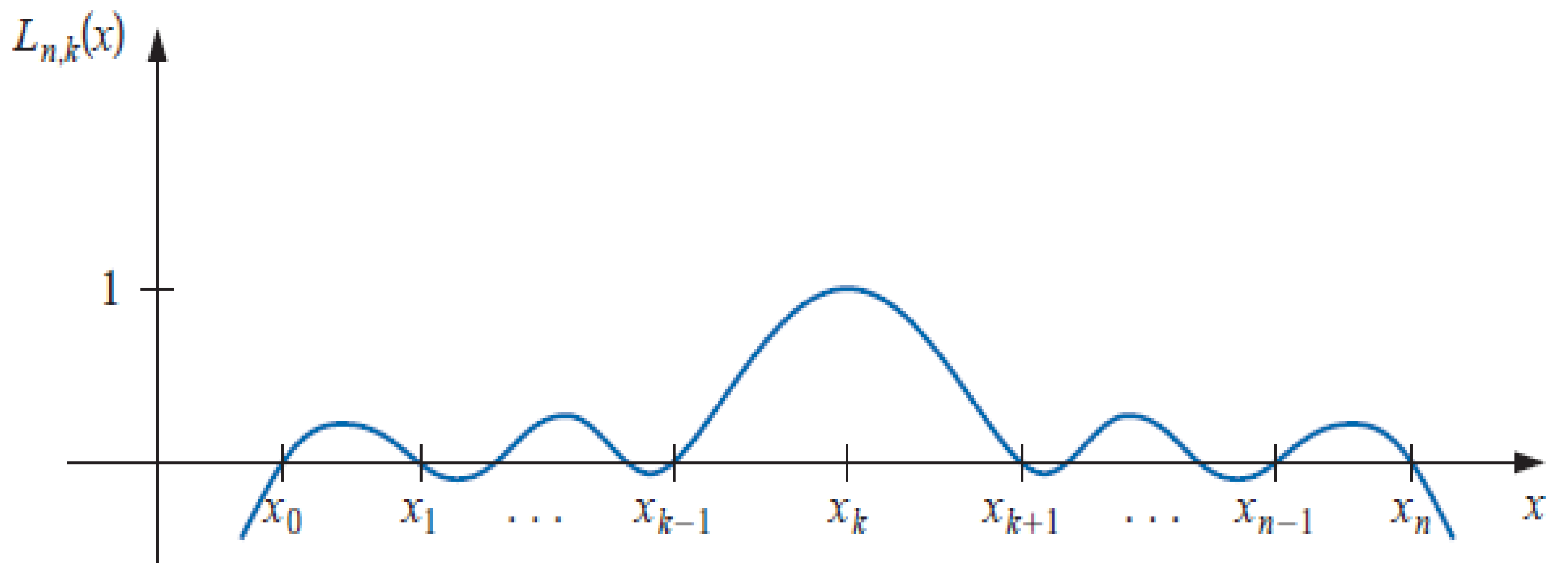
$$(x - x_0)(x - x_1) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n).$$

To satisfy $L_{n,k}(x_k) = 1$, the denominator of $L_{n,k}(x)$ must be this same term but evaluated at $x = x_k$. Thus

$$L_{n,k}(x) = \frac{(x - x_0) \cdots (x - x_{k-1})(x - x_{k+1}) \cdots (x - x_n)}{(x_k - x_0) \cdots (x_k - x_{k-1})(x_k - x_{k+1}) \cdots (x_k - x_n)}.$$

A sketch of the graph of a typical $L_{n,k}$ (when $n$ is even) is shown in Figure 3.5.

Figure 3.5



**Example 2** **(a)** Use the numbers (called *nodes*) $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$ to find the second Lagrange interpolating polynomial for $f(x) = 1/x$.

**(b)** Use this polynomial to approximate $f(3) = 1/3$.

***Solution*** **(a)** We first determine the coefficient polynomials $L_0(x)$, $L_1(x)$, and $L_2(x)$. In nested form they are

$$L_0(x) = \frac{(x-2.75)(x-4)}{(2-2.5)(2-4)} = \frac{2}{3}(x-2.75)(x-4),$$

$$L_1(x) = \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} = -\frac{16}{15}(x-2)(x-4),$$

and

$$L_2(x) = \frac{(x-2)(x-2.75)}{(4-2)(4-2.5)} = \frac{2}{5}(x-2)(x-2.75).$$

Also, $f(x_0) = f(2) = 1/2$, $f(x_1) = f(2.75) = 4/11$, and $f(x_2) = f(4) = 1/4$, so

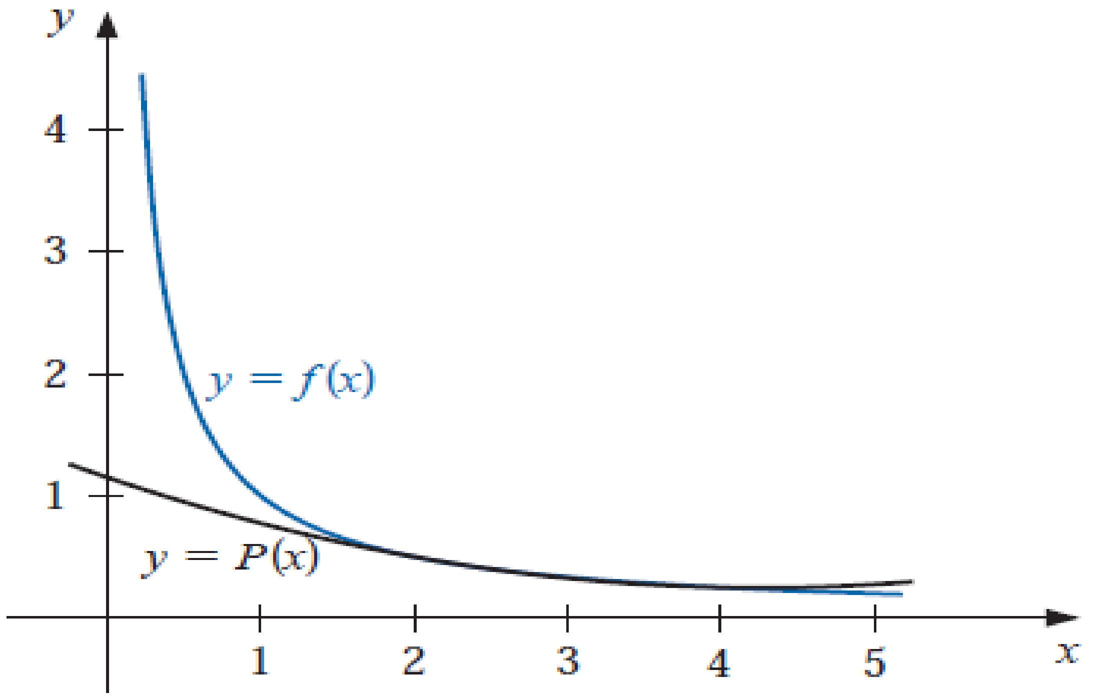
$$\begin{aligned}
P(x) &= \sum_{k=0}^{2} f(x_k)L_k(x) \\
&= \frac{1}{3}(x - 2.75)(x - 4) - \frac{64}{165}(x - 2)(x - 4) + \frac{1}{10}(x - 2)(x - 2.75) \\
&= \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}.
\end{aligned}$$

**(b)** An approximation to $f(3) = 1/3$ (see Figure 3.6) is

$$f(3) \approx P(3) = \frac{9}{22} - \frac{105}{88} + \frac{49}{44} = \frac{29}{88} \approx 0.32955.$$

Recall that in the opening section of this chapter (see Table 3.1) we found that no Taylor polynomial expanded about $x_0 = 1$ could be used to reasonably approximate $f(x) = 1/x$ at $x = 3$. ■

**Figure 3.6**



# Errors of Newton's Interpolating Polynomials/

- Structure of interpolating polynomials is similar to the Taylor series expansion in the sense that finite divided differences are added sequentially to capture the higher order derivatives.
- For an $n^{th}$-order interpolating polynomial, an analogous relationship for the error is:

$$R_n = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\cdots(x-x_n)$$

$\xi$ Is somewhere containing the unknown and he data

- For non differentiable functions, if an additional point $f(x_{n+1})$ is available, an alternative formula can be used that does not require prior knowledge of the function:

$$R_n \cong f[x_{n+1}, x_n, x_{n-1}, \ldots, x_0](x-x_0)(x-x_1)\cdots(x-x_n)$$

# Lagrange Interpolating Polynomials



- The Lagrange interpolating polynomial is simply a reformulation of the Newton’s polynomial that avoids the computation of divided differences:

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$f_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

$$f_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

- As with Newton's method, the Lagrange version has an estimated error of:

$$R_n = f[x, x_n, x_{n-1}, \ldots, x_0] \prod_{i=0}^{n} (x - x_i)$$

**Example 3** In Example 2 we found the second Lagrange polynomial for $f(x) = 1/x$ on $[2, 4]$ using the nodes $x_0 = 2$, $x_1 = 2.75$, and $x_2 = 4$. Determine the error form for this polynomial, and the maximum error when the polynomial is used to approximate $f(x)$ for $x \varepsilon [2, 4]$.

***Solution*** Because $f(x) = x^{-1}$, we have

$$f'(x) = -x^{-2}, \quad f''(x) = 2x^{-3}, \quad \text{and} \quad f'''(x) = -6x^{-4}.$$

As a consequence, the second Lagrange polynomial has the error form

$$\frac{f'''(\xi(x))}{3!}(x-x_0)(x-x_1)(x-x_2) = -(\xi(x))^{-4}(x-2)(x-2.75)(x-4), \quad \text{for } \xi(x) \text{ in } (2, 4).$$

The maximum value of $(\xi(x))^{-4}$ on the interval is $2^{-4} = 1/16$. We now need to determine the maximum value on this interval of the absolute value of the polynomial

$$g(x) = (x-2)(x-2.75)(x-4) = x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22.$$

Because

$$D_x\left(x^3 - \frac{35}{4}x^2 + \frac{49}{2}x - 22\right) = 3x^2 - \frac{35}{2}x + \frac{49}{2} = \frac{1}{2}(3x-7)(2x-7),$$

the critical points occur at

$$x = \frac{7}{3}, \text{ with } g\left(\frac{7}{3}\right) = \frac{25}{108}, \qquad \text{and} \qquad x = \frac{7}{2}, \text{ with } g\left(\frac{7}{2}\right) = -\frac{9}{16}.$$

Hence, the maximum error is

$$\frac{f'''(\xi(x))}{3!}|(x-x_0)(x-x_1)(x-x_2)| \le \frac{1}{16\cdot 6}\left|-\frac{9}{16}\right| = \frac{3}{512} \approx 0.00586. \quad \blacksquare$$