2.2 CHEBYSHEV POLYNOMIALS

The solution of the differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2y = 0 \tag{2.19}$$

is called the *Chebyshev polynomial.*

The Chebyshev polynomial of first kind is $T_n(x)$.

$$T_n(x) = y = \cos(n \cos^{-1} x)$$

To verify, $y = \cos (n \cos^{-1} x)$ satisfies the differential equation (2.19)

$$y = \cos (n \cos^{-1} x)$$

$$\frac{dy}{dx} = \frac{n}{\sqrt{(1-x^2)}} \sin (n \cos^{-1} x)$$

$$\sqrt{(1-x^2)}\frac{dy}{dx} = n \sin(n \cos^{-1} x)$$

Differentiating both sides with respect to x, we have

$$\sqrt{(1-x^2)}\frac{d^2y}{dx^2} - \frac{x}{\sqrt{(1-x^2)}}\frac{dy}{dx} = -n^2 \frac{\cos(n \cos x)}{\sqrt{(1-x^2)}}$$

Multiplying both sides by $\sqrt{(1-x^2)}$, we get

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = -n^2 \cos(n \cos^{-1} x)$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0$$

Therefore, $y = T_n(x) = \cos(n \cos^{-1} x)$ is the solution of the differential equation (2.19).

The **Chebyshev polynomial of second kind** is $U_n(x)$.

To verify, $y = \sin(n \cos^{-1} x)$ satisfies the differential equation (2.19).

$$U_n(x) = y = \sin(n \cos^{-1} x)$$

$$\frac{dy}{dx} = -\frac{n}{\sqrt{(1-x^2)}} \cos(n \cos^{-1} x)$$

$$\sqrt{(1-x^2)}\frac{dy}{dx} = -n \cos(n \cos^{-1} x)$$

Differentiating both sides with respect to x, we get

$$\sqrt{(1-x^2)}\frac{d^2y}{dx^2} - \frac{x}{\sqrt{(1-x^2)}}\frac{dy}{dx} = -n^2 \frac{\sin(n \cos^{-1} x)}{\sqrt{(1-x^2)}}$$

Multiplying both sides by $\sqrt{(1-x^2)}$, we have

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0$$

Chebyshev polynomials

$$n = 0,\ T_0(x) = \cos 0 = 1$$

$$T_1(x) = \cos(\cos^{-1} x) = x$$

$$T_2(x) = \cos(2\theta), \text{ where } \theta = \cos^{-1} x,\ x = \cos\theta$$

$$= 2\cos^2\theta - 1 = 2x^2 - 1$$

$$T_3(x) = \cos 3\theta = 4\cos^3\theta - 3\cos\theta = 4x^3 - 3x$$

$$T_4(x) = \cos 4\theta = 2\cos^2 2\theta - 1 = 2(2x^2 - 1)^2 - 1$$

$$= 8x^4 - 8x^2 + 1$$

$T_5(x) = \cos 5\theta = \cos 4\theta \cos\theta - \sin 4\theta \sin\theta$

$= \cos 4\theta \cos\theta - \sin\theta \cdot 2\cos 2\theta \sin 2\theta$

$= x(8x^4 - 8x^2 + 1) - \sqrt{(1-x^2)}\ 2 \cdot 2x\sqrt{(1-x^2)}\,(2x^2 - 1)$

$= 16x^5 - 20x^3 + 5x$

$T_6(x) = \cos 6\theta = 2\cos^2 3\theta - 1$

$= 2(4x^3 - 3x)^2 - 1$

$= 32x^6 - 48x^4 + 18x^2 - 1$

$T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$

$T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$

$T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$

$U_0(x) = \sin 0 = 0$

$U_1(x) = \sin(\cos^{-1}x) = \sin\theta = \sqrt{(1-x^2)}$

$U_2(x) = \sin(2\theta)$, where $\theta = \cos^{-1}x$, $x = \cos\theta$

$= 2\sin\theta\cos\theta = 2x\sqrt{(1-x^2)}$

$U_3(x) = \sin 3\theta = 3\sin\theta - 4\sin^3\theta = \sin\theta(3 - 4\sin^2\theta)$

$= \sqrt{(1-x^2)}\,(4x^2 - 1)$

$U_4(x) = \sin 4\theta = 2\cos 2\theta \sin 2\theta = \sqrt{(1-x^2)}\ (8x^3 - 4x)$

Notes:

1. $T_n(1) = \cos 0 = 1$
2. $T_n(0) = \cos\dfrac{n\pi}{2}$ = if n is odd

   $= (-1)^{n/2}$, if n is even
3. $T_n(-1) = (-1)^n$
4. $U_n(1) = \sin 0 = 0$
5. $U_n(-1) = \sin\pi = 0$
6. $U_n(0) = \sin\dfrac{n\pi}{2} = 0$ if n is even

   $= (-1)^n$ if n is odd

2.2.1 Series Solution

Suppose $y = \sum_0^{\infty} a_r x^{k-r}$

Differentiating with respect to x, one gases

$$\frac{dy}{dx} = \sum_0^{\infty} a_r (k-r) x^{k-r-1}$$

Differentiating with respect to x again, one gets

$$\frac{d^2y}{dx^2}=\sum_0^\infty a_r(k-r)(k-r-1)x^{k-r-2}$$

Substituting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in Eq. (2.19)

$$(1-x^2)\sum_0^\infty a_r(k-r)(k-r-1)x^{k-r-2}-x\sum_0^\infty a_r(k-r)x^{k-r-1}+n^2\sum_0^\infty a_r x^{k-r}=0$$

$$\sum_0^\infty a_r(k-r)(k-r-1)x^{k-r-2}-\sum_0^\infty a_r[(k-r)(k-r-1)+(k-r)-n^2]x^{k-r}=0$$

$$\sum_0^\infty a_r(k-r)(k-r-1)x^{k-r-2}-\sum_0^\infty a_r[(k-r)^2-n^2]x^{k-r}=0$$

Comparing the coefficients of x^k, we have

$$a_0(k^2-n^2)=0,\ k=n,-n$$

Comparing the coefficients of x^{k-1}, we get

$$a_1((k-1)^2-n^2)=0,\ k\neq 1+n,\neq 1-n.\ \text{So, } a_1=0.$$

Comparing the coefficients of x^{k-r-2}, we have

$$a_r(k-r)(k-r-1)=a_{r+2}((k-r-2)^2-n^2)$$

$$a_{r+2}=\frac{(k-r)(k-r-1)a_r}{[(k-r-2)^2-n^2]}$$

But $k=n$

$$a_{r+2}=\frac{(n-r)(n-r-1)a_r}{[(n-r-2)^2-n^2]}=\frac{(n-r)(n-r-1)a_r}{[(r+2)^2-2n(r+2)]}$$

When $r=0$,

$$a_2=\frac{n(n-1)}{2\cdot(2-2n)}a_0$$

$$a_3=\frac{(n-1)(n-2)}{3\cdot(3-2n)}a_1=0$$

$$a_{2n+1}=0,\ n=0,1,2,3\ \ldots$$

$$a_4=\frac{(n-3)(n-2)}{4\cdot(4-2n)}\frac{n(n-1)}{2\cdot(2-2n)}a_0$$

$$a_6=\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot4\cdot6(2-2n)(4-2n)(6-2n)}a_0$$

$$T_n(x) = a_0\left(x^n - \frac{n(n-1)}{2\cdot(2n-2)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-2)(2n-4)}x^{n-4} - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 4\cdot 6(2n-2n)(2n-4)(2n-6)}x^{n-6} + \cdots\right)$$

Taking $a_0 = 2^{n-1}$,

$$T_n(x) = \frac{n}{2}\left[\frac{(2x)^n}{n} - (2x)^{n-2} + \frac{(n-3)!\,(2x)^{n-4}}{2(n-4)!} - \frac{(n-4)!(2x)^{n-6}}{3!(n-6)!} + \cdots\right]$$

$$T_n(x) = \frac{n}{2}\sum_{0}^{N}\frac{(-1)^r(n-r-1)!}{r!(n-2r)!}(2x)^{n-2r}$$

$$N = \frac{n}{2}, \text{ if } n \text{ is even}$$

$$= \frac{(n-1)}{2}, \text{ if } n \text{ is odd}$$

Matrix representation

$T_n(x)$ satisfies the detetrminant

$$\begin{vmatrix} x & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2x & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2x & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2x & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2x & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2x & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2x \end{vmatrix}$$

For $n = 2$

$$T_2(x) = \begin{vmatrix} x & 1 \\ 1 & 2x \end{vmatrix} = 2x^2 - 1$$

For $n = 3T_3(x) = \begin{vmatrix} x & 1 & 0 \\ 1 & 2x & 1 \\ 0 & 1 & 2x \end{vmatrix} = 4x^3 - 3x$

For $n = 4T_4(x) = \begin{vmatrix} x & 1 & 0 & 0 \\ 1 & 2x & 1 & 0 \\ 0 & 1 & 2x & 1 \\ 0 & 0 & 1 & 2x \end{vmatrix}$

$$= x(8x^3 - 4x) - (4x^2 - 1) = 8x^4 - 8x^2 + 1$$

2.2.2 Generating Function

The function which generates the Chebyshev polynomials is

$$\frac{(1-zx)}{(1-2xz+z^2)} \sum_{0}^{\infty} z^n T_n(x) = \frac{(1-zx)}{(1-2xz+z^2)},$$

i.e. the coefficient of z^n in the expansion of $\dfrac{(1-zx)}{(1-2xz+z^2)}$ is $T_n(x)$.

Proof:

$$\begin{aligned}
\frac{(1-zx)}{(1-2xz+z^2)} &= (1-zx)[1-z(2x-z)]^{-1} \\
&= (1-zx)[1+z(2x-z)+z^2(2x-z)^2+z^3(2x-z)^3+z^4(2x-z)^4 \ldots \\
&= (1-zx)[1+z(2x-z)+z^2(4x^2-4xz+z^2) \\
&\qquad + z^3(8x^3-12x^2z+6xz^2-z^3)+\ldots \\
&= 1+zx+(2x^2-1)z^2+(4x^3-3x)z^3+(8x^4-8x^2+1)z^4+\cdots \\
&= T_0(x)+T_1(x)z+T_2(x)z^2+T_3(x)z^3+T_4(x)z^4+T_5(x)z^5+\cdots \\
&= \sum_{0}^{\infty} z^n T_n(x)
\end{aligned}$$

Therefore, $\dfrac{(1-zx)}{(1-2xz+z^2)}$ is the generating function of $T_n(x)$.

2.2.3 Orthogonality of $T_n(x)$

(i) $$\int_{-1}^{1} \frac{T_n(x)T_m(x)dx}{\sqrt{1-x^2}} = 0 \text{ if } m \neq n$$

(ii) $$\int_{-1}^{1} \frac{T_n(x)T_m(x)dx}{\sqrt{1-x^2}} = \frac{\pi}{2} \text{ if } m = n \neq 0$$

(iii) $$\int_{-1}^{1} \frac{T_n(x)T_m(x)dx}{\sqrt{1-x^2}} = \pi,\ m = n = 0$$

Proof:

(i) $$\int_{-1}^{1} \frac{T_n(x)T_m(x)dx}{\sqrt{1-x^2}} = \int_{-1}^{1} \frac{\cos(n\cos^{-1}x)\cos(m\cos^{-1}x)dx}{\sqrt{1-x^2}},\ m \neq n$$

Put $\cos^{-1} x = \theta$, $\cos\theta = x$, $-\dfrac{dx}{\sqrt{1-x^2}} = d\theta$

When $x = -1$, $\theta = \pi$, when $x = 1$, $\theta = 0$

$$\int_{-1}^{1} \frac{\cos(n\cos^{-1}x)\cos(m\cos^{-1}x)dx}{\sqrt{1-x^2}} = -\int_{\pi}^{0} \cos n\theta \cos m\theta d\theta$$

$$= \int_{0}^{\pi} [\cos(n+m)\theta + \cos(n-m)\theta] d\theta$$

$$= \left[\frac{\sin(n+m)\theta}{(n+m)} + \frac{\sin(n-m)\theta}{(n-m)}\right]_0^{\pi} = 0$$

(ii) $\displaystyle\int_{-1}^{1} \frac{T_n(x)T_n(x)dx}{\sqrt{1-x^2}}$, $m = n$

$$= \int_{-1}^{1} \frac{\cos(n\cos^{-1}x)\cos(n\cos^{-1}x)dx}{\sqrt{1-x^2}}$$

Put $\cos^{-1} x = \theta$, $\cos\theta = x$, $-\dfrac{dx}{\sqrt{1-x^2}} = d\theta$

When $x = -1$, $\theta = \pi$, when $x = 1$, $\theta = 0$

$$\int_{-1}^{1} \frac{\cos(n\cos^{-1}x)\cos(n\cos^{-1}x)dx}{\sqrt{1-x^2}} = \int_{0}^{\pi} \cos^2 n\theta d\theta = \frac{\pi}{2}$$

(iii) $$\int_{-1}^{1} \frac{T_0^{\,2}(x)dx}{\sqrt{1-x^2}} = \int_{-1}^{1} \frac{dx}{\sqrt{1-x^2}} = 2\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}} = \pi$$

Orthogonality of $U_n(x)$

(i) $\displaystyle\int_{-1}^{1} \frac{U_n(x)U_m(x)dx}{\sqrt{1-x^2}} = 0$, if $m \neq n$

(ii) $\displaystyle\int_{-1}^{1} \frac{T_n(x)T_m(x)dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$, if $m = n \neq 0$

(iii) $\displaystyle\int_{-1}^{1} \frac{T_n(x)T_m(x)dx}{\sqrt{1-x^2}} = 0$, if $m = n = 0$

Proof:

(i) $$\int_{-1}^{1}\frac{U_n(x)U_m(x)dx}{\sqrt{1-x^2}}=\int_{-1}^{1}\frac{\sin(n\cos^{-1}x)\sin(m\cos^{-1}x)dx}{\sqrt{1-x^2}}, m\neq n$$

Put $\cos^{-1}x=\theta$, $\cos\theta=x$, $-\dfrac{dx}{\sqrt{1-x^2}}=d\theta$

When $x=-1$, $\theta=\pi$, when $x=1$, $\theta=0$

$$\int_{-1}^{1}\frac{\sin(n\cos^{-1}x)\sin(m\cos^{-1}x)dx}{\sqrt{1-x^2}}=-\int_{\pi}^{0}\sin n\theta\sin m\theta d\theta$$

$$=\int_{0}^{\pi}[\cos(n-m)\theta-\cos(n+m)\theta]d\theta$$

$$=\left[\frac{\sin(n-m)}{(n-m)}-\frac{\sin(n+m)\theta}{(n+m)}\right]_0^{\pi}=0$$

(ii) $$\int_{-1}^{1}\frac{U_n^{\prime}(x)U_n(x)dx}{\sqrt{1-x^2}}, m=n$$

$$=\int_{-1}^{1}\frac{\sin(n\cos^{-1}x)\sin(n\cos^{-1}x)dx}{\sqrt{1-x^2}}$$

Put $\cos^{-1}x=\theta$, $\cos\theta=x$, $-\dfrac{dx}{\sqrt{1-x^2}}=d\theta$

When $x=-1$, $\theta=\pi$, when $x=1$, $\theta=0$

$$\int_{-1}^{1}\frac{\sin(n\cos^{-1}x)\sin(n\cos^{-1}x)dx}{\sqrt{1-x^2}}=\int_{0}^{\pi}\sin^2 n\theta d\theta$$

$$=\frac{\pi}{2}$$

(iii) $$\int_{-1}^{1}\frac{U_0^{\,2}(x)dx}{\sqrt{1-x^2}}=0$$

2.2.4 Recurrence Relations

1. $T_{n+1}(x)+T_{n-1}(x)=2xT_n(x)$

Proof:

$$T_n(x) = \cos(n \cos^{-1} x)$$

Put $\cos^{-1} x = \theta, \cos\theta = x$

$$T_n(x) = \cos(n\theta)$$

$$T_{n+1}(x) = \cos(n+1)\theta$$

$$= \cos n\theta \cos\theta - \sin n\theta \sin\theta$$

$$T_{n-1}(x) = \cos(n-1)\theta$$

$$= \cos n\theta \cos\theta + \sin n\theta \sin\theta$$

Adding $T_{n+1}(x) + T_{n-1}(x) = 2\cos n\theta \cos\theta$

$$= 2x\, T_n(x) \qquad [\cos\theta = x]$$

$\therefore$ $T_{n+1}(x) + T_{n-1}(x) = 2x\, T_n(x)$

2. $(1 - x^2)\, T_n'(x) = nT_{n-1}(x) - nx\, T_n(x)$

Proof:

$$T_n(x) = \cos(n \cos^{-1} x)$$

Put $\cos^{-1} x = \theta, \cos\theta = x$

$$T_n(x) = \cos(n\theta)$$

$$T_n'(x) = \frac{n \sin n\theta}{\sqrt{1-x^2}}$$

Multiplying both sides by $\sqrt{(1-x^2)}$, we get

$$\sqrt{(1-x^2)}\; T_n'(x) = n \sin n\theta$$

Multiplying both sides by $\sqrt{(1-x^2)}$, we have

$$(1 - x^2)\, T_n'(x) = \sqrt{(1-x^2)}\; n \sin n\theta$$

$$= n \sin\theta \sin n\theta$$

Consider $T_{n-1}(x) = \cos(n-1)\theta$

$$= \cos n\theta \cos\theta + \sin n\theta \sin\theta$$

$$\sin\theta \sin n\theta = T_{n-1}(x) - \cos n\theta \cos\theta$$

$$= T_{n-1}(x) - xT_n(x)$$

$\therefore$ $(1 - x^2)\, T_n'(x) = n(T_{n-1}(x) - xT_n(x))$

$$(1 - x^2)\, T_n'(x) = nT_{n-1}(x) - nxT_n(x)$$

3. $U_{n+1}(x) + U_{n-1}(x) = 2xU_n(x)$

Proof:

$$U_n(x) = \sin(n \cos^{-1} x)$$

Put
$$\cos^{-1} x = \theta,\ \cos\theta = x$$
$$U_n(x) = \sin(n\theta)$$
$$U_{n+1}(x) = \sin(n+1)\theta$$
$$= \sin n\theta \cos\theta + \cos n\theta \sin\theta$$
$$U_{n-1}(x) = \sin(n-1)\theta$$
$$= \sin n\theta \cos\theta - \cos n\theta \sin\theta$$

Adding
$$T_{n+1}(x) + T_{n-1}(x) = 2\sin n\theta \cos\theta$$
$$= 2xU_n(x) \qquad [\cos\theta = x]$$

$\therefore$
$$U_{n+1}(x) + U_{n-1}(x) = 2xU_n(x)$$

4.
$$(1 - x^2)U_n'(x) = nxU_n(x) - nU_{n-1}(x)$$

Proof:
$$U_n(x) = \sin(n\cos^{-1} x)$$

Put
$$\cos^{-1} x = \theta,\ \cos\theta = x$$
$$U_n(x) = \sin(n\theta)$$
$$U_n'(x) = \frac{n\cos n\theta}{\sqrt{1-x^2}}$$

Multiplying both sides by $\sqrt{(1-x^2)}$, we have
$$\sqrt{(1-x^2)}\,U_n'(x) = n\cos n\theta$$

Multiplying both sides by $\sqrt{(1-x^2)}$, we have
$$(1 - x^2)U'(x) = \sqrt{(1-x^2)}\,n\cos n\theta$$
$$= n\sin\theta\cos n\theta$$

Consider
$$U_{n-1}(x) = \sin(n-1)\theta$$
$$= \sin n\theta\cos\theta - \cos n\theta\sin\theta$$
$$\sin\theta\cos n\theta = \sin n\theta\cos\theta - U_{n-1}(x)$$
$$= xU_n(x) - U_{n-1}(x)$$

$\therefore$
$$(1 - x^2)U_n'(x) = nxU_n(x) - U_{n-1}(x)$$
$$(1 - x^2)U'(x) = nxU_n(x) - nU_{n-1}(x)$$

Polynomials in terms of $T_n(x)$

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$x^2 = \frac{T_2(x) + T_0(x)}{2}$$

$$T_3(x) = 4x^3 - 3x$$

$$x^3 = \frac{T_3(x)+3T_1(x)}{4}$$

$$T_4(x) = 8x^4 - 8x^2 + 1$$

$$x^4 = \frac{T_4(x)+4T_2(x)+3T_0(x)}{8}$$

WORKED-OUT PROBLEMS

1. Prove that $T_n(-x) = (-1)^n T_n(x)$

Solution:

$$T_n(x) = \cos(n \cos^{-1} x)$$

$$T_n(-x) = \cos(n \cos^{-1} x)$$

$$= \cos(n(\pi - \theta)$$

$$= \cos n\pi \cos n\theta = (-1)^n T_n(x)$$

2. Express $x^2 - x + 1$ in terms of Chebyshev polynomials.

Solution:

$$x^2 - x + 1 = \frac{T_2(x)+T_0(x)}{2} - T_1(x) + T_0(x)$$

$$= \frac{T_2(x)-2T_1(x)+3T_0(x)}{2}$$

3. Express $x^3 + 4x^2 - 3x + 5$ in terms of Chebyshev polynomials.

Solution:

$$x^3 = \frac{T_3(x)+3T_1(x)}{4}$$

$$x^2 = \frac{T_2(x)+T_0(x)}{2}$$

$$x = T_1(x),\ 1 = T_0(x)$$

$$x^3 + 4x^2 - 3x + 5 = \frac{T_3(x)+3T_1(x)}{4} + 4\frac{T_2(x)+T_0(x)}{2}$$

$$-3T_1(x),\ + 5T_0(x)$$

$$= \frac{T_3(x)+8T_2(x)-9T_1(x)+13T_0(x)}{4}$$

4. Express $8x^4 - 8x^3 - 6x^2 + 5x - 4$ in terms of Chebyshev polynomials

Solution:

$$8x^4 = T_4(x) + 4T_2(x) + 3T_0(x)$$

$$8x^3 = 2T_3(x) + 6T_1(x)$$

$$6x^2 = 3T_2(x) + 3T_0(x)$$

$$x = T_1(x),\ 1 = T_0(x)$$

$$8x^4 - 8x^3 - 6x^2 + 5x - 4 = T_4(x) + 4T_2(x) + 3T_0(x) - (2T_3(x) + 6T_1(x))$$
$$+ 3T_2(x) + 3T_0(x) + 5T_1(x) - 4T_0(x)$$
$$= T_4(x) - 2T_3(x) + 7T_2(x) - T_1(x) + 2T_0(x)$$

5. Prove that $\int_{-1}^{1} \frac{x^3 T_4(x)(dx)}{\sqrt{1-x^2}} = 0$

Solution:
$$= \int_{-1}^{1} \frac{x^3 \cos(4\cos^{-1} x)(dx)}{\sqrt{1-x^2}}$$

Put $\cos^{-1} x = \theta$, $\cos\theta = x$

$$d\theta = -\frac{dx}{\sqrt{1-x^2}}$$

$$= \int_{0}^{\pi} \cos^3\theta \cos 4\theta d\theta$$

$$= 0 \quad [\cos(\pi - \theta) = -\cos\theta, \cos(4(\pi - \theta)) = \cos 4\theta]$$

6. Write $T_2(x) + T_1(x) + T_0(x)$ as a polynomial.

Solution:
$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_2(x) + T_1(x) + T_0(x) = 2x^2 + x$$

7. Write $T_3(x) - 4T_2(x) + 5T_1(x) - 6T_0(x)$ as a polynomial

Solution:
$$T_3(x) = 4x^3 - 3x,\ T_2(x) = 2x^2 - 1$$
$$T_1(x) = x,\ T_0(x) = 1$$
$$T_3(x) - 4T_2(x) + 5T_1(x) - 6T_0(x)$$
$$= 4x^3 - 3x - 4(2x^2 - 1) + 5x - 6$$
$$= 4x^3 - 8x^2 + 2x - 2$$

8. Evaluate $\int_{-1}^{1} x^2 T_2(x) dx$

Solution: Put $\cos^{-1} x = \theta$, $\cos\theta = x$

$$d\theta = \frac{dx}{\sqrt{1-x^2}},\ dx = \sin\theta\, d\theta$$

$$\int_{-1}^{1} x^2 T_2(x)dx = \int_0^{\pi} \cos^2\theta \cos 2\theta \sin\theta d\theta$$

$$= \int_0^{\pi} \cos^2\theta(2\cos^2\theta - 1)\sin\theta d\theta$$

$$= 2\int_0^{\pi/2} \cos^2\theta(2\cos^2\theta - 1)\sin\theta d\theta$$

$$= 4\int_0^{\pi/2} \cos^4\theta \sin\theta d\theta - 2\int_0^{\pi/2} \cos^2\theta \sin\theta d\theta$$

$$= 4\cdot\frac{1}{5} - 2\cdot\frac{1}{3} = \frac{2}{15}$$

9. Evaluate $\int_{-1}^{1} \frac{T_4(x)T_6(x)dx}{\sqrt{1-x^2}}$

Solution: $\int_{-1}^{1} \frac{T_4(x)T_6(x)dx}{\sqrt{1-x^2}} = 0$ by orthogonality $(m \neq n)$

10. Evaluate $\int_{-1}^{1} \frac{[T_5(x)]^2 dx}{\sqrt{1-x^2}}$

Solution: $\int_{-1}^{1} \frac{[T_5(x)]^2 dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$ by orthogonality

11. Evaluate $\int_0^1 x^3(1-x^2)^{1/2} T_2(x)dx$

Solution: $\int_0^1 x^3(1-x^2)^{1/2} T_2(x)dx = \int_0^1 x^3(1-x^2)^{1/2} \cos(2\cos^{-1}x)dx$

Put $\cos^{-1} x = \theta$, $\cos\theta = x$,

$$d\theta = \frac{dx}{\sqrt{1-x^2}},\ dx = \sin\theta\, d\theta \sqrt{(1-x^2)} = \sin\theta$$

$$= \int_0^{\pi/2} \cos^3\theta \sin^2\theta \cos 2\theta d\theta$$

$$\int_0^{\pi/2} \cos^3\theta \sin^2\theta(2\cos^2\theta - 1)d\theta = 2\int_0^{\pi/2} \cos^5\theta \sin^2\theta d\theta - \int_0^{\pi/2} \cos^3\theta \sin^2\theta d\theta$$

$$= \frac{16}{105} - \frac{2}{15} = \frac{2}{105}$$

12. Prove that $T_{3n}(x) = 2T_n(x)\,T_{2n}(x) - T_n(x)$

$$T_{2n}(x) = \cos 2n\theta = 2\cos^2 n\theta - 1 = 2[T_{2n}]^2 - 1$$

$$T_{3n}(x) = \cos 3n\theta = 4\cos^3 n - 3\cos n\theta$$

$$= \cos n\theta\,(4\cos^2 n\theta - 3) = T_n(x)[2(T_{2n} + 1) - 3$$

$$= 2T_n(x)\,T_{2n}(x) - T_n(x)$$

13. Evaluate $\int_0^1 x^5(1-x^2)^{3/2}T_3(x)dx$

Solution: $\int_0^1 x^5(1-x^2)^{3/2}T_3(x)dx$

Put $\cos^{-1} x = \theta$, $\cos\theta = x$,

$$d\theta = \frac{dx}{\sqrt{1-x^2}},\ dx = \sin\theta\, d\theta\sqrt{(1-x^2)} = \sin\theta$$

$$\int_0^1 x^5(1-x^2)^{3/2}T_3(x)dx = \int_0^{\pi/2} \cos^5\theta \sin^3\theta \cos 3\theta \sin\theta d\theta$$

$$= \int_0^{\pi/2} \cos^5\theta \sin^4\theta(4\cos^3\theta - 3\cos\theta)d\theta$$

$$= 4\int_0^{\pi/2} \cos^8\theta \sin^4\theta d\theta - 3\int_0^{\pi/2} \cos^6\theta \sin^4\theta d\theta$$

$$= \frac{7\pi}{2^9} - \frac{9\pi}{2^9} = -\frac{\pi}{2^8}$$

14. Evaluate $\int_{-1}^1 x^4(1-x^2)^{-1/2}T_2(x)dx$

Solution: $\int_{-1}^1 x^4(1-x^2)^{-1/2}\,T_2(x)dx$

Put $\cos^{-1} x = \theta$, $\cos\theta = x$,

$$d\theta = \frac{dx}{\sqrt{1-x^2}},$$

$$dx = \sin\theta\, d\theta \sqrt{(1-x^2)} = \sin\theta$$

$$\int_{-1}^{1} x^4 (1-x^2)^{-1/2} T_2(x)dx = \int_0^{\pi} \cos^4\theta \cos 2\theta d\theta$$

$$= 2\int_0^{\pi/2} \cos^4\theta (2\cos^2\theta - 1)d\theta$$

$$= 4\int_0^{\pi/2} \cos^6\theta d\theta - 2\int_0^{\pi/2} \cos^4\theta d\theta$$

$$= \frac{5\pi}{8} - \frac{3\pi}{8} = \frac{\pi}{4}$$

15. Evaluate $\int_0^1 x(1-x^2)^{-1/2} U_4(x)dx$

Solution: $\int_0^1 x(1-x^2)^{-1/2} U_4(x)dx$

Put $\cos^{-1} x = \theta$, $\cos\theta = x$,

$$d\theta = -\frac{dx}{\sqrt{1-x^2}},$$

$$dx = -\sin\theta\, d\theta, \sqrt{(1-x^2)} = \sin\theta$$

$$\int_0^1 x(1-x^2)^{-1/2} U_4(x)dx = \int_0^{\pi/2} \cos\theta \sin 4\theta d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} (\sin 5\theta + \sin 3\theta)d\theta$$

$$= -\frac{1}{2}\left(\frac{\cos 5\theta}{5} + \frac{\cos 3\theta}{3}\right)_0^{\pi/2} = \frac{4}{15}$$

16. Evaluate $\int_{-1}^1 x^2 U_3(x)dx$

Solution: $\int_{-1}^1 x^2 U_3(x)dx$

Put $\cos^{-1} x = \theta$, $\cos\theta = x$,

$$d\theta = -\frac{dx}{\sqrt{1-x^2}},\ dx = -\sin\theta\, d\theta,\ \sqrt{(1-x^2)} = \sin\theta$$

$$\int_{-1}^{1} x^2 U_3(x)dx = \int_0^{\pi} \cos^2\theta \sin 3\theta \sin\theta d\theta$$

$$2\int_0^{\pi/2} \cos^2\theta \sin\theta\,(3\sin\theta - 4\sin^3\theta)d\theta = 6\cdot\frac{\pi}{16} - 8\cdot\frac{\pi}{32} = \frac{\pi}{8}$$

17. Prove that $U_{n+1}(x) = xU_n + \sqrt{(1-x^2)}\,T_n(x)$.

Solution: $U_{n+1}(x) = \sin(n+1)\theta = \sin n\theta \cos\theta + \cos n\theta \sin\theta$

$$= xU_n + \sqrt{(1-x^2)}\,T_n(x)$$