Chapter 3

EQUILIBRIUM OF RIGID BODIES

3-1 Introduction

When the force and the couple are both equal to zero, the external forces form a system equivalent to zero, and the rigid body is said to be in **equilibrium**. We can obtain the necessary and sufficient conditions for the equilibrium of a rigid body by setting **R** and M_O^R equal to zero.

$$\Sigma \mathbf{F} = 0 \qquad \Sigma \mathbf{M}_O = \Sigma (\mathbf{r} \times \mathbf{F}) = 0 \tag{3-1}$$

Resolving each force and each moment into its rectangular components, we can replace these vector equations for the equilibrium of a rigid body with the following six scalar equations:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma F_z = 0 \tag{3-2}$$

$$\Sigma M_x = 0 \qquad \Sigma M_y = 0 \qquad \Sigma M_z = 0 \tag{3-3}$$

In order to write the equations of equilibrium for a rigid body, we must first identify all of the forces acting on that body and then draw the corresponding **free-body diagram**.

Free-Body Diagrams

In solving a problem concerning a rigid body in equilibrium, it is essential to consider *all* of the forces acting on the body. It is equally important to exclude any force that is *not* directly applied to the body. Omitting a force or adding an extraneous one would destroy the conditions of equilibrium. Therefore, the first step in solving the problem is to draw a **free-body diagram** of the rigid body under consideration.

We summarize here the steps you must follow in drawing a correct free-body diagram.

1. Start with a clear decision regarding the choice of the free body to be analyzed. Mentally, you need to detach this body from the ground and separate it from all other bodies. Then you can sketch the contour of this isolated body.

2. Indicate all external forces on the free-body diagram. These forces represent the actions exerted *on* the free body *by* the ground and *by* the bodies that have been detached. In the diagram, apply these forces at the various points where the free body was supported by the ground or was connected to the other bodies. Generally, you should include the *weight* of the free body among the external forces, since it represents the attraction exerted by the earth on the various particles forming the free body. If the free body is made of several parts, do *not* include the forces the various parts exert on each other among the external forces. These forces are internal forces as far as the free body is concerned.

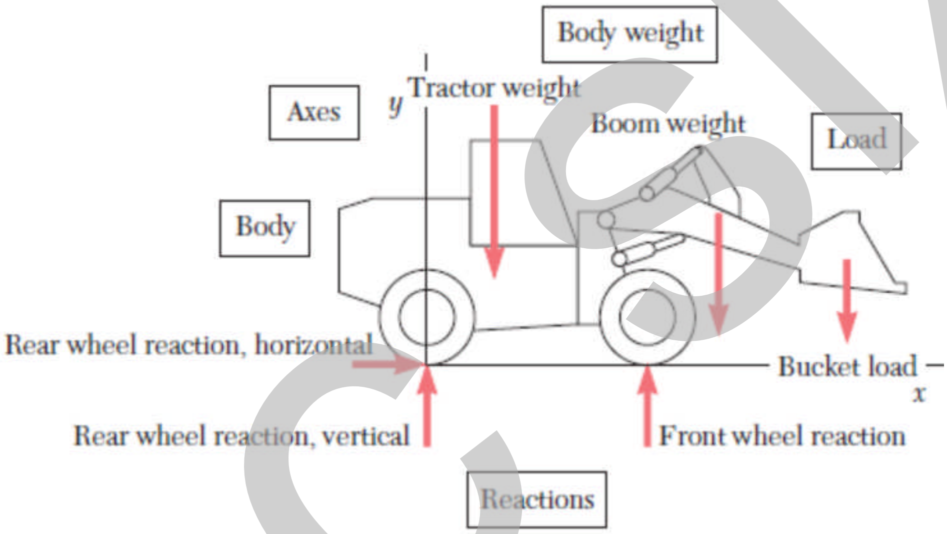


Fig. 3- 1A tractor supporting a bucket load. As shown, its free-body diagram should include all external forces acting on the tractor.

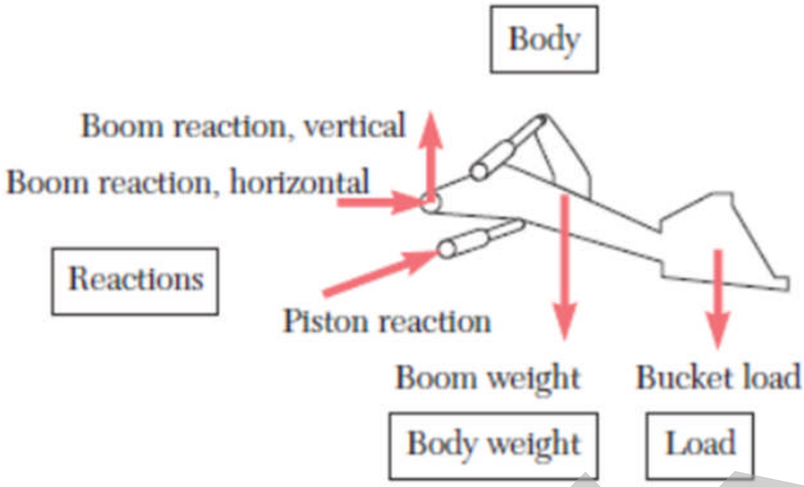


Fig. 3-2 Tractor bucket and boom. The internal forces associated with interconnected members.

3. Clearly mark the magnitudes and directions of the *known external forces* on the free-body diagram. Recall that when indicating the directions of these forces, the forces are those exerted *on*, and not *by*, the free body. Known external forces generally include the *weight* of the free body and *forces applied* for a given purpose.

4. *Unknown external forces* usually consist of the **reactions** through which the ground and other bodies oppose a possible motion of the free body. The reactions constrain the free body to remain in the same position; for that reason, they are sometimes called *constraining forces*. Reactions are exerted at the points where the free body is *supported by* or *connected to* other bodies; you should clearly indicate these points.

5. The free-body diagram should also include dimensions, since these may be needed for computing moments of forces. Any other detail, however, should be omitted.

3-2 Equilibrium in Two Dimensions

3-2-1 Reactions for a Two-Dimensional Structure

The reactions exerted on a two-dimensional structure fall into three categories that correspond to three types of supports or connections.

1. Reactions Equivalent to a Force with a Known Line of Action. Supports and connections causing reactions of this type include rollers, rockers, frictionless surfaces, short links and cables, collars on frictionless rods, and frictionless pins in slots. Each of these supports and connections can prevent motion in one direction only. Figure 3.3 shows these supports and connections together with the reactions they produce. Each reaction involves one unknown—specifically, the magnitude of the reaction. In problem solving, you should denote this magnitude by an appropriate letter. The line of action of the

reaction is known and should be indicated clearly in the free-body diagram. Assume that single-track rollers and rockers are reversible, so the corresponding reactions can be directed either way.

2. Reactions Equivalent to a Force of Unknown Direction and Magnitude. Supports and connections causing reactions of this type include frictionless pins in fitted holes, hinges, and rough surfaces. They can prevent translation of the free body in all directions, but they cannot prevent the body from rotating about the connection. Reactions of this group involve two unknowns and are usually represented by their x and y components. In the case of a rough surface, the component normal to the surface must be directed away from the surface.
3. Reactions Equivalent to a Force and a Couple. These reactions are caused by fixed supports that oppose any motion of the free body and thus constrain it completely. Fixed supports actually produce forces over the entire surface of contact; these forces, however, form a system that can be reduced to a force and a couple. Reactions of this group involve three unknowns usually consisting of the two components of the force and the moment of the couple.

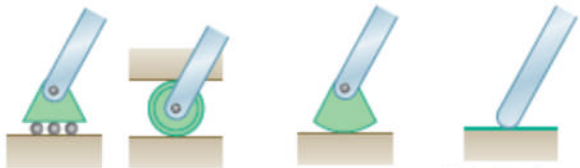
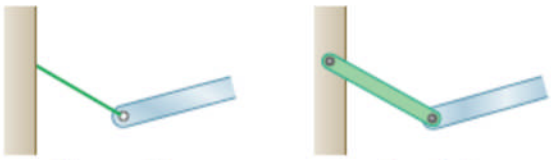
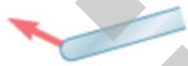
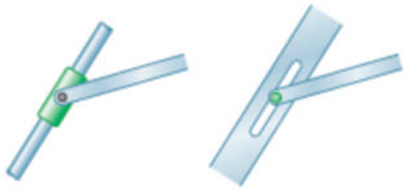
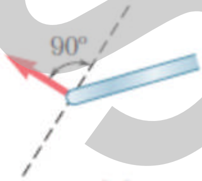
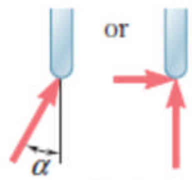
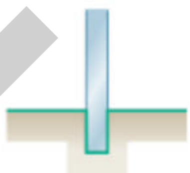
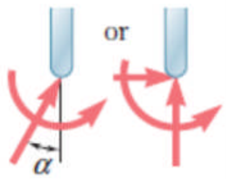
Support or Connection	Reaction	Number of Unknowns
 Rollers; Rocker; Frictionless surface	 Force with known line of action perpendicular to surface	1
 Short cable; Short link	 Force with known line of action along cable or link	1
 Collar on frictionless rod; Frictionless pin in slot	 Force with known line of action perpendicular to rod or slot	1
 Frictionless pin or hinge; Rough surface	 Force of unknown direction	2
 Fixed support	 Force and couple	3

Fig 3-3 Reactions of supports and connections in two dimensions.

Rigid-Body Equilibrium in Two Dimensions

The conditions stated for the equilibrium of a rigid body become considerably simpler for the case of a two-dimensional structure. Choosing the *x* and *y* axes to be in the plane of the structure, we have

$$F_z = 0 \qquad M_x = M_y = 0 \qquad M_z = M_O$$

For each of the forces applied to the structure. Thus, the six equations of equilibrium reduce to three equations:

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_O = 0 \tag{3. 4}$$

Since $\Sigma M_O = 0$ must be satisfied regardless of the choice of the origin O, we can write the equations of equilibrium for a two-dimensional structure in the more general form

Equations of equilibrium in two dimensions

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_A = 0 \tag{3. 5}$$

Where *A* is any point in the plane of the structure. These three equations can be solved for no more than *three unknowns.*

A correct free-body diagram is essential for the successful solution of a problem. Never proceed with the solution of a problem until you are sure that your free-body diagram includes all loads, all reactions, and the weight of the body (if appropriate).

1. You can write three equilibrium equations and solve them for *three unknowns.*

The three equations might be

You have just seen that unknown forces include reactions and that the number of unknowns corresponding to a given reaction depends upon the type of support or connection causing that reaction. Referring to Fig. 3.3, note that you can use the equilibrium equations (3.5) to determine the reactions associated with two rollers and one cable, or one fixed support, or one roller and one pin in a fitted hole, etc.

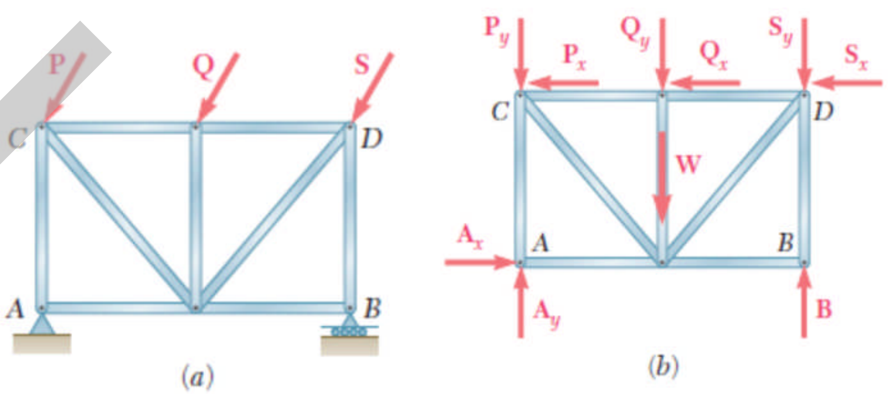


Fig. 3-4 *(a)* A truss supported by a pin and a roller; *(b)* free-body diagram of the truss.

For a two-dimensional rigid body, the reactions at the supports can involve one, two, or three unknowns, depending on the type of support (Fig. 4.1). A correct free-body diagram is essential for the successful solution of a problem. Never proceed with the solution of a problem until you are sure that your free-body diagram includes all loads, all reactions, and the weight of the body (if appropriate).

1. You can write three equilibrium equations and solve them for *three unknowns.* The three equations might be

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0 \qquad \Sigma M_O = 0$$

However, usually several alternative sets of equations are possible, such as

$$\Sigma F_x = 0 \qquad \Sigma M_A = 0 \qquad \Sigma M_B = 0$$

where point B is chosen in such a way that the line AB is not parallel to the y axis, or

$$\Sigma M_A = 0 \qquad \Sigma M_B = 0 \qquad \Sigma M_C = 0$$

where the points A, B, and C do not lie along a straight line.

2. To simplify your solution, it may be helpful to use one of the following solution techniques.

a. By summing moments about the point of intersection of the lines of action of two unknown forces, you obtain an equation in a single unknown.

b. By summing components in a direction perpendicular to two unknown parallel forces, you also obtain an equation in a single unknown.

3. After drawing your free-body diagram, you may find that one of the following special situations arises.

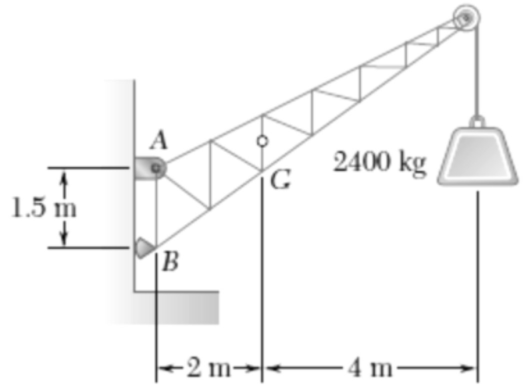
a. The reactions involve fewer than three unknowns. The body is said to be **partially constrained** and motion of the body is possible.

b. The reactions involve more than three unknowns. The reactions are said to be **statically indeterminate.** Although you may be able to calculate one or two reactions, you cannot determine all of them.

c. The reactions pass through a single point or are parallel. The body is said to be **improperly constrained** and motion can occur under a general loading condition.

EXAMPLE 3-1

A fixed crane has a mass of 1000 kg and is used to lift a 2400-kg crate. It is held in place by a pin at A and a rocker at B. The center of gravity of the crane is located at G. Determine the components of the reactions at A and B.



STRATEGY: Draw a free-body diagram to show all of the forces acting on the crane, then use the equilibrium equations to calculate the values of the unknown forces.

MODELING:

Free-Body Diagram. By multiplying the masses of the crane and of the crate by $g = 9.81 \text{ m/s}^2$, you obtain the corresponding weights—that is, 9810 N or 9.81 kN, and 23 500 N or 23.5 kN (Fig. 1). The reaction at pin A is a force of unknown direction; you can represent it by components $\mathbf{A}_x$ and $\mathbf{A}_y$. The reaction at the rocker B is perpendicular to the rocker surface; thus, it is horizontal. Assume that $\mathbf{A}_x$, $\mathbf{A}_y$, and $\mathbf{B}$ act in the directions shown.

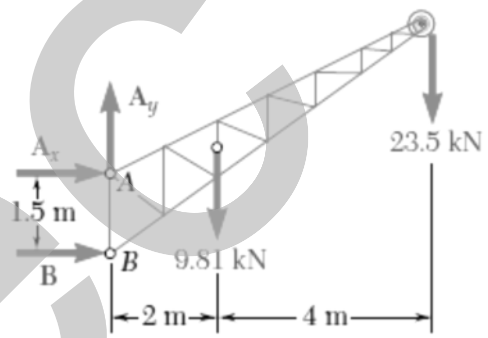


Fig. 1 Free-body diagram of crane.

ANALYSIS:

Determination of *B*. The sum of the moments of all external forces about point A is zero. The equation for this sum contains neither A_x nor A_y, since the moments of $\mathbf{A}_x$ and $\mathbf{A}_y$ about A are zero. Multiplying the magnitude of each force by its perpendicular distance from A, you have

$$+\circlearrowleft \Sigma M_A = 0: \qquad +B(1.5 \text{ m}) - (9.81 \text{ kN})(2 \text{ m}) - (23.5 \text{ kN})(6 \text{ m}) = 0$$

$$B = +107.1 \text{ kN} \qquad \mathbf{B} = 107.1 \text{ kN} \rightarrow$$

Since the result is positive, the reaction is directed as assumed.

Determination of $\mathbf{A}_x$. Determine the magnitude of $\mathbf{A}_x$ by setting the sum of the horizontal components of all external forces to zero.

$$\xrightarrow{+} \Sigma F_x = 0: \qquad A_x + B = 0$$

$$A_x + 107.1 \text{ kN} = 0$$

$$A_x = -107.1 \text{ kN} \qquad \mathbf{A}_x = 107.1 \text{ kN} \leftarrow$$

Since the result is negative, the sense of $\mathbf{A}_x$ is opposite to that assumed originally.

Determination of $\mathbf{A}_y$. The sum of the vertical components must also equal zero. Therefore,

$$+\uparrow \Sigma F_y = 0: \qquad A_y - 9.81 \text{ kN} - 23.5 \text{ kN} = 0$$

$$A_y = +33.3 \text{ kN} \qquad \mathbf{A}_y = 33.3 \text{ kN} \uparrow$$

Adding the components $\mathbf{A}_x$ and $\mathbf{A}_y$ vectorially, you can find that the reaction at A is 112.2 kN $\nwarrow$17.3°.

REFLECT and THINK: You can check the values obtained for the reactions by recalling that the sum of the moments of all the external forces about any point must be zero. For example, considering point B (Fig. 2), you can show

$$+\circlearrowleft \Sigma M_B = -(9.81 \text{ kN})(2 \text{ m}) - (23.5 \text{ kN})(6 \text{ m}) + (107.1 \text{ kN})(1.5 \text{ m}) = 0$$

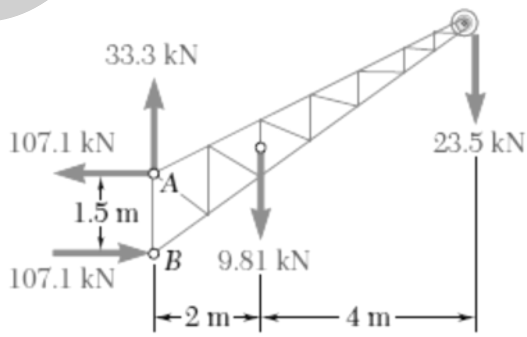
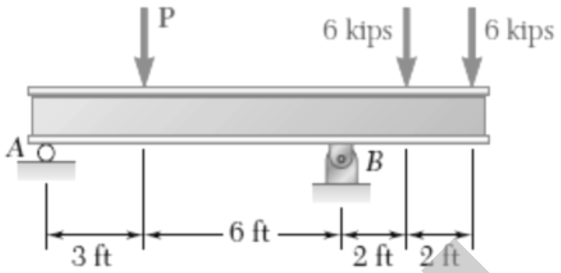


Fig. 2 Free-body diagram of crane with solved reactions.

EXAMPLE 3-2

Three loads are applied to a beam as shown. The beam is supported by a roller at *A* and by a pin at *B*. Neglecting the weight of the beam, determine the reactions at *A* and *B* when $P = 15$ kips.



STRATEGY: Draw a free-body diagram of the beam, then write the equilibrium equations, first summing forces in the *x* direction and then summing moments at *A* and at *B*.

MODELING:

Free-Body Diagram. The reaction at *A* is vertical and is denoted by **A** (Fig. 1). Represent the reaction at *B* by components $\mathbf{B}_x$ and $\mathbf{B}_y$. Assume that each component acts in the direction shown.

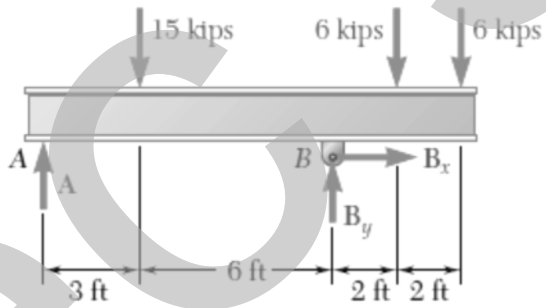


Fig.1 Free-body diagram of beam.

ANALYSIS:

Equilibrium Equations. Write the three equilibrium equations and solve for the reactions indicated:

$\overset{+}{\rightarrow}\Sigma F_x = 0$: $\quad B_x = 0 \qquad \mathbf{B}_x = 0$ ◀

$+\circlearrowleft\Sigma M_A = 0$:

$$-(15 \text{ kips})(3 \text{ ft}) + B_y(9 \text{ ft}) - (6 \text{ kips})(11 \text{ ft}) - (6 \text{ kips})(13 \text{ ft}) = 0$$

$$B_y = +21.0 \text{ kips} \qquad \mathbf{B}_y = 21.0 \text{ kips}\uparrow$$ ◀

$+\circlearrowleft \Sigma M_B = 0$:

$$-A(9 \text{ ft}) + (15 \text{ kips})(6 \text{ ft}) - (6 \text{ kips})(2 \text{ ft}) - (6 \text{ kips})(4 \text{ ft}) = 0$$

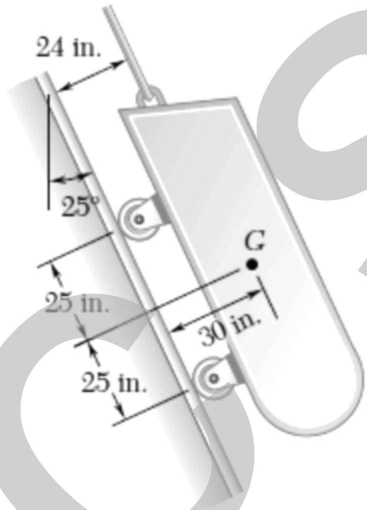
$$A = +6.00 \text{ kips} \qquad \mathbf{A} = 6.00 \text{ kips} \uparrow$$

REFLECT and THINK: Check the results by adding the vertical components of all of the external forces:

$$+\uparrow \Sigma F_y = +6.00 \text{ kips} - 15 \text{ kips} + 21.0 \text{ kips} - 6 \text{ kips} - 6 \text{ kips} = 0$$

EXAMPLE 3-3

A loading car is at rest on a track forming an angle of 25° with the vertical. The gross weight of the car and its load is 5500 lb, and it acts at a point 30 in. from the track, halfway between the two axles. The car is held by a cable attached 24 in. from the track. Determine the tension in the cable and the reaction at each pair of wheels.



STRATEGY: Draw a free-body diagram of the car to determine the unknown forces, and write equilibrium equations to find their values, summing moments at A and B and then summing forces.

MODELING:

Free-Body Diagram. The reaction at each wheel is perpendicular to the track, and the tension force **T** is parallel to the track. Therefore, for convenience, choose the x axis parallel to the track and the y axis perpendicular to the track (Fig. 1). Then resolve the 5500-lb weight into x and y components.

$$W_x = +(5500 \text{ lb}) \cos 25° = +4980 \text{ lb}$$
$$W_y = -(5500 \text{ lb}) \sin 25° = -2320 \text{ lb}$$

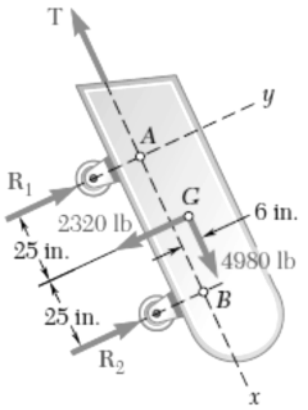


Fig. 1 Free-body diagram of car.

ANALYSIS:

Equilibrium Equations. Take moments about A to eliminate $\mathbf{T}$ and $\mathbf{R}_1$ from the computation.

$$+\circlearrowleft \Sigma M_A = 0: \qquad -(2320 \text{ lb})(25 \text{ in.}) - (4980 \text{ lb})(6 \text{ in.}) + R_2(50 \text{ in.}) = 0$$
$$R_2 = +1758 \text{ lb} \qquad \mathbf{R}_2 = 1758 \text{ lb} \nearrow \quad \blacktriangleleft$$

Then take moments about B to eliminate $\mathbf{T}$ and $\mathbf{R}_2$ from the computation.

$$+\circlearrowleft \Sigma M_B = 0: \qquad (2320 \text{ lb})(25 \text{ in.}) - (4980 \text{ lb})(6 \text{ in.}) - R_1(50 \text{ in.}) = 0$$
$$R_1 = +562 \text{ lb} \qquad \mathbf{R}_1 = +562 \text{ lb} \nearrow \quad \blacktriangleleft$$

Determine the value of T by summing forces in the x direction.

$$\searrow + \Sigma F_x = 0: \qquad +4980 \text{ lb} - T = 0$$
$$T = +4980 \text{ lb} \qquad \mathbf{T} = 4980 \text{ lb} \nwarrow \quad \blacktriangleleft$$

Figure 2 shows the computed values of the reactions.

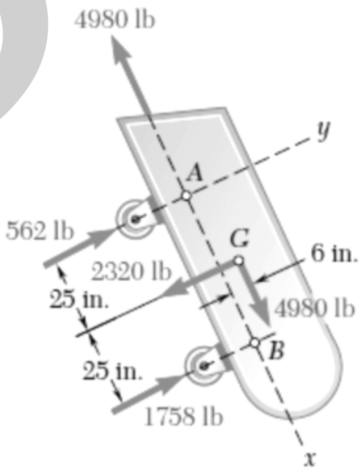


Fig. 2 Free-body diagram of car with solved reactions.

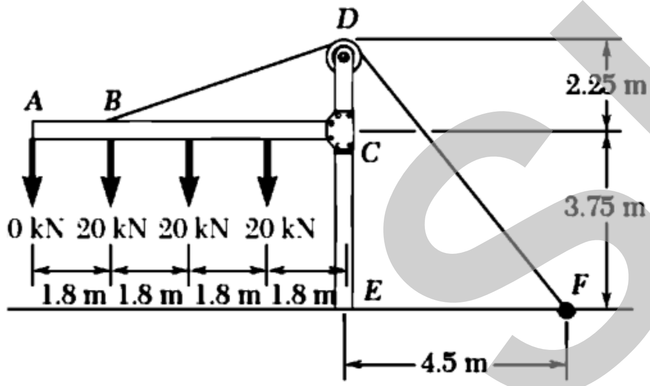
REFLECT and THINK: You can verify the computations by summing forces in the y direction.

$$\nearrow + \Sigma F_y = +562 \text{ lb} + 1758 \text{ lb} - 2320 \text{ lb} = 0$$

You could also check the solution by computing moments about any point other than A or B.

EXAMPLE 3-4

The frame shown supports part of the roof of a small building. Knowing that the tension in the cable is 150 kN, determine the reaction at the fixed end E.



STRATEGY: Draw a free-body diagram of the frame and of the cable BDF. The support at E is fixed, so the reactions here include a moment; to determine its value, sum moments about point E.

MODELING:

Free-Body Diagram. Represent the reaction at the fixed end E by the force components $\mathbf{E}_x$ and $\mathbf{E}_y$ and the couple $\mathbf{M}_E$ (Fig. 1). The other forces acting on the free body are the four 20-kN loads and the 150-kN force exerted at end F of the cable.

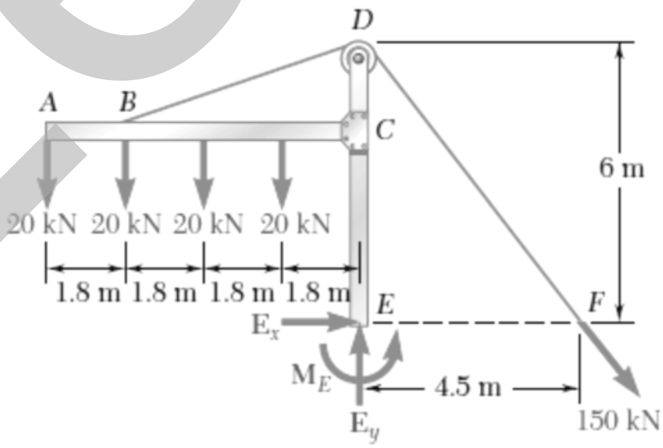


Fig. 1 Free-body diagram of frame.

ANALYSIS:

Equilibrium Equations. First note that

$$DF = \sqrt{(4.5 \text{ m})^2 + (6 \text{ m})^2} = 7.5 \text{ m}$$

Then you can write the three equilibrium equations and solve for the reactions at E.

$$\xrightarrow{+}\Sigma F_x = 0: \qquad E_x + \frac{4.5}{7.5}(150 \text{ kN}) = 0$$

$$E_x = -90.0 \text{ kN} \qquad \mathbf{E}_x = 90.0 \text{ kN} \leftarrow$$

$$+\uparrow\Sigma F_y = 0: \qquad E_y - 4(20 \text{ kN}) - \frac{6}{7.5}(150 \text{ kN}) = 0$$

$$E_y = +200 \text{ kN} \qquad \mathbf{E}_y = 200 \text{ kN} \uparrow$$

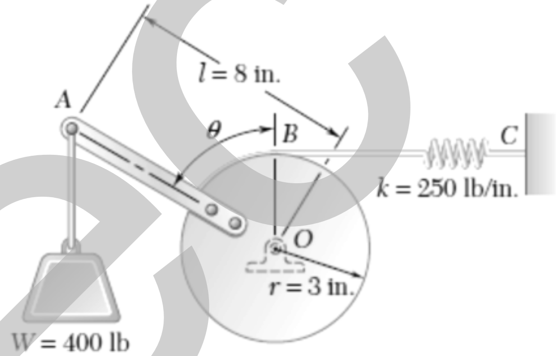
$$+\circlearrowleft\Sigma M_E = 0: \qquad (20 \text{ kN})(7.2 \text{ m}) + (20 \text{ kN})(5.4 \text{ m}) + (20 \text{ kN})(3.6 \text{ m})$$

$$+(20 \text{ kN})(1.8 \text{ m}) - \frac{6}{7.5}(150 \text{ kN})(4.5 \text{ m}) + M_E = 0$$

$$M_E = +180.0 \text{ kN}\cdot\text{m} \qquad \mathbf{M}_E = 180.0 \text{ kN}\cdot\text{m} \circlearrowleft$$

EXAMPLE 3-5

A 400-lb weight is attached at A to the lever shown. The constant of the spring BC is $k = 250$ lb/in., and the spring is unstretched when $\theta = 0$. Determine the position of equilibrium.



STRATEGY: Draw a free-body diagram of the lever and cylinder to show all forces acting on the body (Fig. 1), then sum moments about O. Your final answer should be the angle θ.

MODELING:

Free-Body Diagram. Denote by s the deflection of the spring from its unstretched position and note that $s = r\theta$. Then $F = ks = kr\theta$.

ANALYSIS:

Equilibrium Equation. Sum the moments of **W** and **F** about O to eliminate the reactions supporting the cylinder. The result is

$$+\circlearrowleft\Sigma M_O = 0: \qquad Wl \sin\theta - r(kr\theta) = 0 \qquad \sin\theta = \frac{kr^2}{Wl}\theta$$

Substituting the given data yields

$$\sin\theta = \frac{(250\text{ lb/in.})(3\text{ in.})^2}{(400\text{ lb})(8\text{ in.})}\theta \quad \sin\theta = 0.703\,\theta$$

Solving by trial and error, the angle is $\theta = 0$ $\qquad \theta = 80.3°$ ◀

REFLECT and THINK: The weight could represent any vertical force acting on the lever. The key to the problem is to express the spring force as a function of the angle θ.

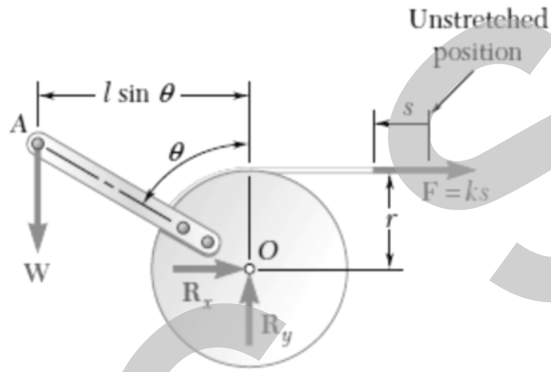


Fig. 1 Free-body diagram of the lever and cylinder.