II.4 Partial Fraction Expansion

1 **a**

$$\frac{x^3}{x^3-4} = 1 + \frac{4}{x^3-4}$$

$$x^3 - 4 \,\overline{)\, x^3 \,}\quad 1, \qquad x^3 - (x^3-4) = 4$$

b

$$\frac{x^3+2x}{x^3-4} = 1 + \frac{2x+4}{x^3-4}$$

$$x^3 - 4 \,\overline{)\, x^3+2x \,}\quad 1, \qquad (x^3+2x) - (x^3-4) = 2x+4$$

c

$$= 1 - \frac{x^2+x+1}{x^3-4}$$

d

$$= x + \frac{x-1}{x^2-1}$$

2 **b**

$$x^2+6x+10 = (x+3)^2 - 9 + 10$$

$$= (x+3)^2 + 1$$

$$\therefore \int \frac{dx}{x^2+6x+10} = \int \frac{dx}{(x+3)^2+1} \longrightarrow \text{Let } u = x+3, \quad du = dx$$

$$= \int \frac{du}{u^2+1} = \tan^{-1} u + c$$

$$= \tan^{-1}(x+3) + c.$$

c) $5x^2+20x+25 = 5(\underbrace{x^2+4x}+5)$

$= 5[(x+2)^2-4+5]$

$= 5[(x+2)^2+1]$

$\therefore \int \frac{dx}{5x^2+20x+25} = \int \frac{dx}{5[(x+2)^2+1]} = \frac{1}{5}\int \frac{dx}{(x+2)^2+1}$

↳ Let $u = x+2 \longrightarrow du = dx$

$= \frac{1}{5}\int \frac{du}{u^2+1} = \frac{1}{5}\tan^{-1}u + c$

$= \frac{1}{5}\tan^{-1}(x+2) + c.$

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3) equating coefficients:-

$$\frac{x^2+3}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

multiplying by $x(x+1)(x-1)$

$\therefore x^2+3 = A(x+1)(x-1) + B(x)(x-1) + C(x)(x+1)$

$\therefore x^2+3 = A(x^2-1) + B(x^2-x) + C(x^2+x)$

$\therefore A + B + C = 1$ —— ①

$-A = 3 \longrightarrow \boxed{A = -3}$ —— ②

$-B + C = 0$ —— ③

$\therefore \boxed{A = -3}$, $\boxed{B = 2}$, $\boxed{C = 2}$

$$\therefore \int \frac{x^2+3}{x(x+1)(x-1)}\,dx = \int \frac{-3}{x}\,dx + \int \frac{2}{x+1}\,dx + \int \frac{2}{x-1}\,dx$$

$$= -3\ln|x| + 2\ln|x+1| + 2\ln|x-1| + c..$$

$\boxed{4}$ Heaviside-trick:-

$$\frac{x^2+3}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}.$$

↳ multiplying by (x):-

$$↳ \frac{\cancel{x}(x^2+3)}{\cancel{x}(x+1)(x-1)} = A + \frac{Bx}{x+1} + \frac{Cx}{x-1}$$

use Limit

$$\therefore A = \frac{x^2+3}{(x+1)(x-1)} - \frac{Bx}{x+1} - \frac{Cx}{x-1}\Bigg|_{\boxed{x=0}}$$

$$\boxed{\therefore A = -3}$$

↳ multiplying by $(x+1)$:-

$$\therefore \frac{x^2+3}{x(x-1)} = \frac{A(x+1)}{x} + B + \frac{c(x+1)}{(x-1)}$$

$$\therefore B = \frac{x^2+3}{x(x-1)} - \frac{A(x+1)}{x} - \frac{c(x+1)}{x-1} \Bigg|_{\boxed{x \to -1}}$$

$$\boxed{\therefore B = 2}$$

↳ multiplying by $(x-1)$:-

$$↳ \frac{x^2+3}{x\cdot(x+1)} = \frac{A(x-1)}{x} + \frac{B(x-1)}{x+1} + C$$

$$\therefore C = \frac{x^2+3}{x(x+1)} - \frac{A(x-1)}{x} - \frac{B(x-1)}{(x+1)} \Bigg|_{\boxed{x \to 1}}$$

$$\boxed{\therefore C = 2}$$

↳ do the same integral as the previous problem.

$\boxed{5}$

$$\hookrightarrow \frac{x^2+3}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$$

$\hookrightarrow$ multiplying by $\underline{x^2(x-1)}$:-

$$\therefore x^2+3 = A(x)(x-1) + B(x-1) + C(x^2)$$

$$\therefore x^2+3 = A(x^2-x) + B(x-1) + Cx^2$$

$$\therefore A+C = 1 \text{ ——— } (1)$$

$$-A+B = 0 \text{ ——— } (2)$$

$$-B = 3 \text{ ——— } (3)$$

$$\therefore \boxed{B=-3} \quad , \quad \boxed{A=-3} \quad , \quad \boxed{C=4}$$

$$\therefore \int \frac{x^2+3}{x^2(x-1)}dx = \int \frac{-3}{x}dx + \int \frac{-3}{x^2}dx + \int \frac{4}{x-1}dx$$

$$= -3\ln|x| + \frac{3}{x} + 4\ln|x-1| + C.$$

$\boxed{7}$

$$\int_{-5}^{-2} \frac{x^4-1}{x^2+1}dx = \int_{-5}^{-2} \frac{\cancel{(x^2+1)}(x^2-1)}{\cancel{(x^2+1)}}dx$$

$$= \int_{-5}^{-2} (x^2-1)\,dx = \frac{x^3}{3} - x \Big|_{-5}^{-2}$$

$$= \underline{36}.$$

[9] $\int \frac{x^5\,dx}{x^2-1} = \int (x^3+x)\,dx + \int \frac{x}{x^2-1}\,dx$

$$\begin{array}{r|l} & x^3+x \\ \hline x^2-1 & x^5 \\ & x^5-x^3 \\ \hline & x^3 \\ & x^3-x \\ \hline & x \end{array}$$

$= \frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{2}\ln|x^2-1| + c.$

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[12] $\int \frac{2x+1}{x^2-3x+2}\,dx \longrightarrow \Rightarrow \frac{2x+1}{x^2-3x+2} = \frac{2x+1}{(x-2)(x-1)}$

$\therefore \frac{2x+1}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$ $[* (x-2)(x-1)]$

$\therefore 2x+1 = A(x-1) + B(x-2)$

$\therefore A+B = 2$ —— (1)

$-A-2B = 1$ —— (2)

$\boxed{A = 5 \quad B = -3}$

$\therefore \int \frac{2x+1}{x^2-3x+2}\,dx = \int \frac{5}{x-2}\,dx + \int \frac{-3}{x-1}\,dx$

$= 5\ln|x-2| - 3\ln|x-1| + C$

$\boxed{14}$ $\displaystyle\int \frac{e^{3x}\,dx}{e^{4x}-1} \longrightarrow$ Let $e^{x} = u \rightarrow du = e^{x}dx$

$$\therefore \int \frac{e^{3x}}{e^{4x}-1}dx = \int \frac{e^{2x} \cdot e^{x}dx}{e^{4x}-1} = \int \frac{u^2\,du}{u^4-1}$$

$$= \int \frac{u^2\,du}{(u^2+1)(u^2-1)} = \int \frac{u^2\,du}{(u^2+1)(u+1)(u-1)}$$

$$\therefore \frac{u^2}{(u^2+1)(u+1)(u-1)} = \frac{A}{u-1} + \frac{B}{u+1} + \frac{Cu+D}{u^2+1}$$

↳ multiplying by $[(u^2+1)(u+1)(u-1)]$:-

$$\therefore u^2 = A(u+1)(u^2+1) + B(u-1)(u^2+1) + [Cu+D(u+1)(u-1)]$$

$$\therefore u^2 = A(u^3+u^2+u+1) + B(u^3-u^2+u-1) + (Cu+D)(u^2-1)$$

$$\therefore \left.\begin{aligned} A+B+C &= 0 \quad \text{— (1)} \\ A-B+D &= 1 \quad \text{— (2)} \\ A+B-C &= 0 \quad \text{— (3)} \\ A-B-D &= 0 \quad \text{— (4)} \end{aligned}\right\} \begin{aligned} &\rightarrow A = \tfrac{1}{4} \\ &\hookrightarrow B = -\tfrac{1}{4} \\ &\hookrightarrow C = 0 \\ &\hookrightarrow D = \tfrac{1}{2} \end{aligned}$$

$$\int \frac{u^2}{(u^2-1)(u^2+1)} = \frac{1}{2}\int \frac{du}{u^2+1} - \frac{1}{4}\int \frac{du}{u+1} + \frac{1}{4}\int \frac{du}{u-1}$$

$$= \frac{1}{2}\tan^{-1}u - \frac{1}{4}\ln|u+1| + \frac{1}{4}\ln|u-1| + c$$

$$= \frac{1}{2}\tan^{-1}e^x - \frac{1}{4}\ln|e^x+1| + \frac{1}{4}\ln|e^x-1| + c.$$

$\boxed{15}$ $\displaystyle\int \frac{e^x\,dx}{\sqrt{1+e^{2x}}} \rightarrow$ Let $u = e^x \rightarrow du = e^x dx.$

$$\hookrightarrow = \int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1}u + c$$

$$= \sinh^{-1}e^x + c.$$

$\boxed{16}$ $\displaystyle\int \frac{e^x\,dx}{e^{2x}+2e^x+2} \longrightarrow e^{2x}+2e^x+2 = (e^x+1)^2 - 1 + 2$

$$= (e^x+1)^2 + 1$$

$\hookrightarrow = \displaystyle\int \frac{e^x\,dx}{(e^x+1)^2+1} \longrightarrow$ Let $u = e^x+1$

$du = e^x dx.$

$$= \int \frac{du}{u^2+1} = \tan^{-1}u + c = \tan^{-1}(e^x+1) + c.$$

$\boxed{17}$ $\int \frac{dx}{1+e^x} \longrightarrow$ Let $u = 1+e^x \rightarrow (e^x = u-1)$

$du = e^x dx$

$du = (u-1)dx$

$\boxed{dx = \frac{du}{u-1}}$

$= \int \frac{du}{(u-1)(u)} \longrightarrow \frac{A}{u} + \frac{B}{u-1} = \frac{1}{u(u-1)}$

$\therefore A(u-1) + B(u) = 1$

$\therefore A + B = 0$ — (1)

$-A = 1$ — (2)

$\left.\right\} \rightarrow \boxed{\begin{matrix} A = -1 \\ B = 1 \end{matrix}}$

$= \int \frac{-1}{u} du + \int \frac{du}{u-1} = -\ln|u| + \ln|u-1| + c$

$= -\ln|e^x+1| + \ln|e^x| + c$. ※

$\boxed{18}$ $\int \frac{dx}{x(x^2+1)} \longrightarrow \frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$\therefore A(x^2+1) + (Bx+C)(x) = 1 \longrightarrow$ Complete to find A, B and C

↳ Solution := $\ln|x| - \frac{1}{2}\ln|x^2+1| + c$ ※

$\boxed{19}$ $\int \frac{dx}{x(x^2+1)^2} \longrightarrow \frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

↳ multiplying by $[x(x^2+1)^2]$

$\therefore A(x^2+1)^2 + (Bx+c)(x)(x^2+1) + (Dx+E)(x) = 1$

$\therefore A(x^4+2x^2+1) + (Bx+c)(x^3+x) + Dx^2 + Ex = 1$

$\therefore A(x^4+2x^2+1) + Bx^4 + Bx^2 + cx^3 + cx + Dx^2 + Ex = 1$

$\therefore A+B=0$ — ①

$C=0$ — ②

$2A+B+D=0$ — ③

$C+E=0$ — ④

$A=1$ — ⑤

$\therefore A=1$

$B=-1$

$C=0$

$D=-1$

$E=0$

$$\therefore \int \frac{dx}{x(x^2+1)^2} = \int \frac{1}{x}dx + \int \frac{-x}{x^2+1}dx + \int \frac{-x}{(x^2+1)^2}$$

$$= \ln|x| - \frac{1}{2}\ln|x^2+1| + \left(\frac{1}{2} \cdot \frac{1}{x^2+1}\right) + c.$$