$\boxed{2.}$ $\int \frac{1}{w+2\sqrt{1-w}+2}\,dw$

Let $u=\sqrt{1-w}$

$u^2=1-w \longrightarrow w=1-u^2$

$\hookrightarrow dw=-2u\,du.$

$$\therefore \int \frac{1}{w+2\sqrt{1-w}+2}\,dw = \int \frac{-2u\,du}{1-u^2+2u+2}$$

$$= \int \frac{-2u\,du}{-u^2+2u+3}$$

$$= \int \frac{2u\,du}{u^2-2u-3}$$

$$\frac{2u}{u^2-2u-3} = \frac{2u}{(u-3)(u+1)} = \frac{A}{u+1}+\frac{B}{u-3}$$

$$\therefore 2u = A(u-3)+B(u+1)$$

get $A=\frac{1}{2}$

$B=\frac{3}{2}$

$$\therefore \int \frac{2u}{u^2-2u-3}\,du = \int \frac{\frac{1}{2}du}{u+1} + \int \frac{\frac{3}{2}du}{u-3}$$

$$= \frac{1}{2}\ln|u+1| + \frac{3}{2}\ln|u-3| + c$$

$$\therefore \int \frac{1}{\omega+2\sqrt{1-\omega}+2}\,d\omega = \frac{1}{2}\ln\left|\sqrt{1-\omega}+1\right| + \frac{3}{2}\ln\left|\sqrt{1-\omega}-3\right| + c.$$

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3. $\int \frac{t-2}{t-3\sqrt{2t-4}+2}\,dt$ $\longrightarrow$ Let $u=\sqrt{2t-4}$

$$u^2 = 2t-4 \longrightarrow \boxed{t = \frac{u^2}{2}+2}$$

$$2u\,du = 2\,dt$$

$$\boxed{\therefore dt = u\,du}$$

$$\therefore \int \frac{t-2 \cdot dt}{t-3\sqrt{2t-4}+2} = \int \frac{\frac{u^2}{2}+2-2}{\frac{u^2}{2}+2-3u+2} \cdot (u)\,du.$$

$$\hookrightarrow = \int \frac{u^3/2}{\frac{u^2}{2}-3u+4}\,du$$

$$= \int \frac{u^3}{u^2-6u+8}\,du.$$

$$= \int (u+6)\,du + \int \frac{28u-48}{u^2-6u+8}\,du$$

$$\begin{array}{r|l} & u+6 \\ \hline u^2-6u+8 & u^3 \\ & u^3-6u^2+8u \\ \hline & 6u^2-8u \\ & 6u^2-36u+48 \\ \hline & 28u-48 \end{array}$$

$$\frac{28u-48}{u^2-6u+8} = \frac{28u-48}{(u-2)(u-4)} = \frac{A}{u-2} + \frac{B}{u-4}$$

$$\therefore A(u-4) + B(u-2) = 28u-48$$

$\hookrightarrow A + B = 28$ — ①

$\hookrightarrow -4A - 2B = -48$ — ②

$\therefore \boxed{A=-4}$, $\boxed{B=32}$

$$\therefore \int (u+6)\,du + \int \frac{28u-48}{u^2-6u+8}\,du =$$

$$= \int (u+6)\,du + \int \frac{-4\,du}{u-2} + \int \frac{32\,du}{u-4}$$

$$= \frac{u^2}{2} + 6u - 4\ln|u-2| + 32\ln|u-4| + C$$

$$= t-2+6\sqrt{2t-4} - 4\ln\left|\sqrt{2t-4}-2\right| + 32\ln\left|\sqrt{2t-4}-4\right| + C$$

$\boxed{4}$ $\int \cos(\sqrt{x})\,dx$

$\hookrightarrow$ Let $u=\sqrt{x}$

$u^2=x$

$\boxed{2u\,du=dx}$

$\hookrightarrow \therefore \int \cos\sqrt{x}\,dx = \int \cos(u)\cdot 2u\,du$

$= 2\int u\cdot\cos(u)\,du$

Let $f=u \longrightarrow f'=du$

$g'=\cos(u)du \longrightarrow g=\sin(u)$

$\therefore \int \cos\sqrt{x}\,dx = 2\int u\cos(u)\,du$

$$= 2\left[u \cdot \sin(u) - \int \sin(u)\,du\right]$$

$$= 2u \cdot \sin(u) + 2\cos(u) + C$$

$$= 2\sqrt{x} \cdot \sin\sqrt{x} + 2\cos\sqrt{x} + c.$$

II.7 Integrals Involving Quadratics

$\boxed{1}$ $\displaystyle\int \frac{7}{\omega^2+3\omega+3}\,d\omega \longrightarrow \omega^2+3\omega+3 = \left(\omega+\frac{3}{2}\right)^2+\frac{3}{4}$

$$= \int \frac{7\,d\omega}{\left(\omega+\frac{3}{2}\right)^2+\frac{3}{4}} \longrightarrow \text{Let } u = \omega+\frac{3}{2}, \quad du = d\omega$$

$$= \int \frac{7\,du}{u^2+\frac{3}{4}} = \int \frac{7\,du}{\frac{3}{4}\left[\left(\frac{2}{\sqrt{3}}u\right)^2+1\right]} = \frac{28}{3}\int \frac{du}{\left(\frac{2}{\sqrt{3}}u\right)^2+1}$$

$$\text{Let } x = \frac{2}{\sqrt{3}}u \rightarrow dx = \frac{2}{\sqrt{3}}\,du \rightarrow du = \frac{\sqrt{3}}{2}\,dx$$

$$= \frac{28}{3}\int \frac{du}{\left(\frac{\sqrt{3}}{2}u\right)^2+1} = \frac{28}{3}\int \frac{\frac{\sqrt{3}}{2}\,dx}{x^2+1} = \frac{14}{\sqrt{3}}\int \frac{dx}{x^2+1}$$

$$= \frac{14}{\sqrt{3}}\tan^{-1}x + c = \frac{14}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}u\right) + c$$

$$= \frac{14}{\sqrt{3}}\tan^{-1}\left[\frac{2\omega+3}{\sqrt{3}}\right] + c \;\#.$$

$\boxed{2}$ $\displaystyle\int \frac{10x}{4x^2-8x+9}\,dx \longrightarrow 4x^2-8x+9 = 4\left(x^2-2x+\frac{9}{4}\right)$

$$= 4\left((x-1)^2+\frac{5}{4}\right)$$

$$= 4(x-1)^2+5$$

$$\hookrightarrow = \int \frac{10x\,dx}{4(x-1)^2+5} \longrightarrow \text{Let } u = x-1$$

$$du = dx$$

$$(x = \underline{\underline{u+1}})$$

$$\hookrightarrow = \int \frac{10(u+1)\,du}{4u^2+5}$$

$$= \int \frac{10u}{4u^2+5}\,du + 10\int \frac{du}{4u^2+5}$$

$$= \int \frac{10u\,du}{4u^2+5} + \frac{10}{5}\int \frac{du}{\left(\frac{2}{\sqrt{5}}u\right)^2+1}$$

$$= \int \frac{10u\,du}{4u^2+5} + 2\int \frac{du}{\left(\frac{2}{\sqrt{5}}u\right)^2+1}$$

$$\text{Let } \frac{2}{\sqrt{5}}u = w \longrightarrow \frac{2}{\sqrt{5}}du = dw$$

$$\therefore du = \frac{\sqrt{5}}{2}dw$$

$$= \int \frac{10u\,du}{4u^2+5} + \cancel{2}\int \frac{\frac{\sqrt{5}}{\cancel{2}}\,dw}{w^2+1}$$

$$= \frac{10}{8}\int \frac{8u\,du}{4u^2+5} + \sqrt{5}\int \frac{dw}{w^2+1}$$

$$= \frac{5}{4}\ln|4u^2+5| + \sqrt{5}\tan^{-1}(w) + c$$

$$= \frac{5}{4}\ln|4(x-1)^2+5| + \sqrt{5}\tan^{-1}\left(\frac{2}{\sqrt{5}}(x-1)\right) + c$$

$$= \frac{5}{4}\ln|4(x-1)^2+5| + \sqrt{5}\tan^{-1}\left[\frac{2x-2}{\sqrt{5}}\right] + c \ \#.$$

$\boxed{3}$ $\displaystyle\int \frac{2t+9}{(t^2-14t+46)^{5/2}}\,dt \longrightarrow t^2-14t+46 = (t-7)^2-3$

$$\hookrightarrow = \int \frac{2t+9\ dt}{((t-7)^2-3)^{5/2}} \longrightarrow \text{Let } \boxed{u = t-7}$$

$$du = dt$$

$$\boxed{t = u+7}$$

$$= \int \frac{2(u+7)+9}{(u^2-3)^{5/2}}\,du = \int \frac{2u+23}{(u^2-3)^{5/2}}\,du$$

$$= \int \frac{2u\,du}{(u^2-3)^{5/2}} + \int \frac{23\,du}{(u^2-3)^{5/2}}$$

$\boxed{\text{the first integral}}$

$$\int \frac{2u\,du}{(u^2-3)^{5/2}} = \int 2u(u^2-3)^{-5/2}\,du$$

$$= (u^2-3)^{-3/2} * \frac{-2}{3}$$

$$= \frac{-2}{3} * \frac{1}{(u^2-3)^{3/2}}$$

$$= \frac{-2}{3} * \frac{1}{((t-7)^2-3)^{3/2}} + c$$

$\boxed{\text{the second integral}}$

$$\int \frac{23\,du}{(u^2-3)^{5/2}}$$

Let $u = \sqrt{3}\sec\theta$

$du = \sqrt{3}\sec\theta\tan\theta\,d\theta$

$$\therefore \int \frac{23\,du}{(u^2-3)^{5/2}} = \int \frac{23 \cdot \sqrt{3}\sec\theta\tan\theta\,d\theta}{(\sqrt{3\sec^2\theta-3})^5}$$

$$= \int \frac{23\sqrt{3}\sec\theta\tan\theta\,d\theta}{(\sqrt{3})^5(\sqrt{\sec^2\theta-1})^5}$$

$$= \frac{23}{9} \int \frac{\sec\theta \tan\theta \, d\theta}{(\sqrt{\tan^2\theta})^5}$$

$$= \frac{23}{9} \int \frac{\sec\theta \tan\theta \, d\theta}{\tan^5\theta}$$

$$= \frac{23}{9} \int \frac{\sec\theta}{\tan^4\theta} d\theta = \frac{23}{9} \int \frac{\cos^3\theta}{\sin^4\theta} d\theta$$

$$= \frac{23}{9} \int \frac{(1-\sin^2\theta)\cos\theta}{\sin^4\theta} d\theta$$

Let $u = \sin\theta$

$du = \cos\theta \, d\theta$

$$= \frac{23}{9} \int \frac{1-u^2}{u^4} du$$

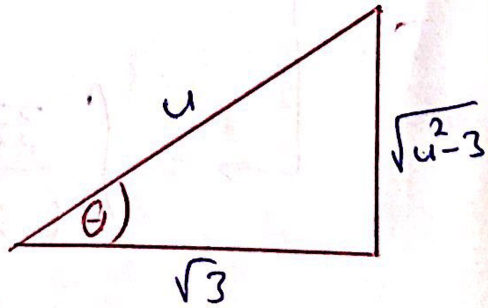
$$= \frac{23}{9} \int \frac{1}{u^4} - \frac{1}{u^2} \, du$$

$$= \frac{23}{9} * \frac{-1}{3} * \frac{1}{u^3} - \frac{23}{9} * -1 * \frac{1}{u}$$

$$= \frac{-23}{27} * \frac{1}{\sin^3\theta} + \frac{23}{9} * \frac{1}{\sin\theta}$$

$\hookrightarrow \because u = \sqrt{3} \sec\theta$

$\sec\theta = \frac{u}{\sqrt{3}}$



$$= -\frac{23}{27} * \frac{u^3}{(\sqrt{u^2-3})^3} + \frac{23}{9} * \frac{u}{\sqrt{u^2-3}} + c$$

$$= \frac{-23}{27} * \frac{(t-7)^3}{(\sqrt{(t-7)^2-3})^3} + \frac{23}{9} * \frac{t-7}{\sqrt{(t-7)^2-3}} + c$$

$$\therefore \int \frac{2t+9}{(t^2-14t+46)^{5/2}} dt = \frac{-2}{3((t-7)^2-3)^{3/2}} - \frac{23(t-7)^3}{27((t-7)^2-3)^{3/2}}$$

$$+ \frac{23(t-7)}{9((t-7)^2-3)^{1/2}} + c$$

$$\boxed{= \frac{23(t-7)}{9\sqrt{(t-7)^2-3}} - \frac{18+23(t-7)^3}{27((t-7)^2-3)^{3/2}} + c} \#.$$

$\boxed{4}$ $\int \frac{3Z}{(1-4Z-2Z^2)^2} dZ \longrightarrow 1-4Z-2Z^2 = -2(Z^2+2Z-\frac{1}{2})$

$$= -2\left[(Z+1)^2 - \frac{3}{2}\right]$$

$$= 3-2(Z+1)^2$$

$$\rightarrow = \int \frac{3Z \cdot dZ}{(3-2(Z+1)^2)^2}$$

let $u = Z+1 \rightarrow Z = u-1 \rightarrow dZ = du$

$$= \int \frac{3(u-1)\,du}{(3-2u^2)^2} = \int \frac{3u-3\,du}{(3-2u^2)^2}$$

$$= \int \frac{3u\,du}{(3-2u^2)^2} - 3\int \frac{du}{(3-2u^2)^2}$$

$\boxed{\text{First integral}}$

$$\hookrightarrow \int \frac{3u\,du}{(3-2u^2)^2} = \frac{-3}{4}\int -4u(3-2u^2)^{-2}\,du$$

$$= -\frac{3}{4} * (3-2u^2)^{-1} * -1$$

$$= \frac{3}{4} * \frac{1}{3-2u^2}$$

$$= \boxed{\frac{3}{4(3-2(z+1)^2)}} + c$$

$\boxed{\text{Second integral}}$

$$\hookrightarrow \int \frac{du}{(3-2u^2)^2}$$

$$\text{Let } u = \frac{\sqrt{3}}{\sqrt{2}} \sin\theta$$

$$du = \frac{\sqrt{3}}{\sqrt{2}} \cos\theta\,d\theta$$

$$2.\ \int \frac{du}{(3-2u^2)^2} = \frac{\sqrt{3}}{\sqrt{2}} \int \frac{\cos\theta\,d\theta}{(3-3\sin^2\theta)^2} = \frac{\sqrt{3}}{\sqrt{2}} \int \frac{\cos\theta\,d\theta}{(3)^2(1-\sin^2\theta)^2}$$

$$= \frac{\sqrt{6}}{18} \int \frac{\cos\theta \cdot d\theta}{(\cos^2\theta)^2}$$

$$= \frac{\sqrt{6}}{18} \int \frac{1}{\cos^3\theta} d\theta = \frac{\sqrt{6}}{18} \int \sec^3\theta \, d\theta$$

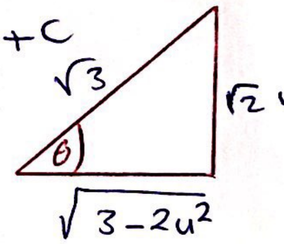
$$= \frac{\sqrt{6}}{18} \int \sec^2\theta \cdot \sec\theta \, d\theta$$

$\boxed{\text{by parts}} \rightarrow u = \sec\theta \rightarrow du = \sec\theta \tan\theta \, d\theta$

$\hookrightarrow dv = \sec^2\theta \, d\theta \rightarrow v = \tan\theta$.

$$\therefore \frac{\sqrt{6}}{18} \int \sec^3\theta = \frac{\sqrt{6}}{2*18} \left[\sec\theta \cdot \tan\theta + \ln|\sec\theta + \tan\theta| \right] + C$$

$$\hookrightarrow = \frac{\sqrt{6}}{36} \cdot \left[\frac{\sqrt{3}}{\sqrt{3-2u^2}} * \frac{\sqrt{2}u}{\sqrt{3-2u^2}} + \ln \left| \frac{\sqrt{3}}{\sqrt{3-2u^2}} + \frac{\sqrt{2}u}{\sqrt{3-2u^2}} \right| \right] + C$$



$$= \boxed{\frac{\sqrt{6}}{36} \left[\frac{\sqrt{6}u}{3-2u^2} + \ln \left| \frac{\sqrt{3}+\sqrt{2}u}{\sqrt{3-2u^2}} \right| + C \right.}$$

$$\boxed{= \frac{\sqrt{6}}{36} \left[\frac{\sqrt{6}(Z+1)}{3-2(Z+1)^2} + \ln \left| \frac{\sqrt{3}+\sqrt{2}(Z+1)}{\sqrt{3-2(Z+1)^2}} \right| + C \right]}$$

$$\therefore \int \frac{3Z \, dZ}{(1-4Z-2Z^2)^2} = \frac{3}{4} * \frac{1}{(3-2(Z+1)^2)} - \frac{3\sqrt{6}}{36} \left[\frac{\sqrt{6}(Z+1)}{3-2(Z+1)^2} \right]$$

$$- \frac{3\sqrt{6}}{36} \ln \left| \frac{\sqrt{3}+\sqrt{2}(Z+1)}{\sqrt{3-2(Z+1)^2}} \right| + C$$

$$= \frac{3}{4(3-2(Z+1)^2)} - \frac{(Z+1)}{2(3-2(Z+1)^2)} - \frac{\sqrt{6}}{12} \ln\left|\frac{\sqrt{3}+\sqrt{2}(Z+1)}{\sqrt{3-2(Z+1)^2}}\right| + c$$

$$= \frac{3}{4(3-2(Z+1)^2)} - \frac{Z+1}{6-4(Z+1)^2} - \frac{1}{2\sqrt{6}} \ln\left|\frac{\sqrt{3}+\sqrt{2}(Z+1)}{\sqrt{3-2(Z+1)^2}}\right| + c$$

$\boxed{6}$ $\displaystyle\int \frac{3-7t}{(t^2+12t+40)^2}\,dt \longrightarrow t^2+12t+40 = (t+6)^2+4$

$$= \int \frac{3-7t}{((t+6)^2+4)^2}\,dt$$

Let $\boxed{t+6=u} \longrightarrow t = u-6$

$dt = du.$

$$= \int \frac{3-7(u-6)\,du}{(u^2+4)^2} = \int \frac{3-7u+42}{(u^2+4)^2}\,du$$

$$= \int \frac{45\,du}{(u^2+4)^2} - 7\int \frac{u}{(u^2+4)^2}\,du$$

$\boxed{\text{First integral}}$

$$\int \frac{45\,du}{(u^2+4)^2} = \int \frac{45\,du}{(4)^2\left(\left(\frac{u}{2}\right)^2+1\right)^2}$$

$$= \frac{45}{16}\int \frac{du}{\left(\left(\frac{u}{2}\right)^2+1\right)^2} \rightarrow \text{Let } x = \frac{u}{2}$$

$$dx = \frac{1}{2}du$$

$$\boxed{du = 2dx}$$

$$= \frac{45}{16}\int \frac{2dx}{(x^2+1)^2} = \frac{45}{8}\int \frac{dx}{(x^2+1)^2}$$

use the reduction formula

$$I_{n+1} = \frac{1}{2n}\cdot\frac{x}{(x^2+1)^n} + \frac{2n-1}{2n} I_n$$

$$I_2 = I_{1+1} = \frac{1}{2}\cdot\frac{x}{x^2+1} + \frac{1}{2} I_1 + c$$

$$= \frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1}x + c$$

$$\hookrightarrow \frac{45}{8}\int \frac{dx}{(x^2+1)^2} = \frac{45}{8}\left[\frac{x}{2(x^2+1)} + \frac{1}{2}\tan^{-1}x + c\right]$$

$$= \frac{45\left(u/2\right)}{16\left(\left(\frac{u}{2}\right)^2+1\right)} + \frac{45}{16}\tan^{-1}\left(\frac{u}{2}\right) + c$$

$$= \frac{45\,(t+6)}{32\left(\left(\frac{t+6}{2}\right)^2+1\right)} + \frac{45}{16}\tan^{-1}\left(\frac{t+6}{2}\right) + c$$

Second integral

$$\hookrightarrow -7\int \frac{u\,du}{(u^2+4)^2} = \frac{-7}{2}\int 2u\,(u^2+4)^{-2}\,du$$

$$= \frac{-7}{2} * (u^2+4)^{-1} * -1 \quad + C$$

$$= \frac{7}{2(u^2+4)} + C = \frac{7}{2((t+6)^2+4)} + C$$

$$\therefore \int \frac{3-7t}{(t^2+12t+40)^2}\,dt = \frac{45(t+6)}{8((t+6)^2+4)} + \frac{45}{16}\tan^{-1}\left(\frac{t+6}{2}\right)$$

$$+ \frac{7}{2((t+6)^2+4)} + C \;\#$$

$$\boxed{= \frac{45(t+6)+28}{8((t+6)^2+4)} + \frac{45}{16}\tan^{-1}\left(\frac{t+6}{2}\right) + C}$$