Lecture 1

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Basic Concepts

- A **differential equation** is any equation which contains derivatives.

Here are a few more examples of differential equations.

$$ay'' + by' + cy = g(t) \quad \textbf{(5)}$$

$$\sin(y)\frac{d^2y}{dx^2} = (1-y)\frac{dy}{dx} + y^2\mathbf{e}^{-5y} \quad \textbf{(6)}$$

$$y^{(4)} + 10y''' - 4y' + 2y = \cos(t) \quad \textbf{(7)}$$

- The **order** of a differential equation is the largest derivative present in the differential equation.
- In the differential equations listed above (5) and (6) are second order differential equations, (7) is a fourth order differential equation.

A **linear differential equation** is any differential equation that can be written in the following form.

$$a_n(t)y^{(n)}(t) + a_{n-1}(t)y^{(n-1)}(t) + \cdots + a_1(t)y'(t) + a_0(t)y(t) = g(t) \quad \textbf{(11)}$$

- Otherwise, it is called a **non-linear** differential equation.
- In (5) - (7) above only (6) is non-linear, the other two are linear differential equations.
- A **solution** to a differential equation on an interval is any function $y(t)$ which satisfies the differential equation in question on that interval.

Example 1 Show that $y(x) = x^{-\frac{3}{2}}$ is a solution to $4x^2y'' + 12xy' + 3y = 0$ for $x > 0$.

First Order Differential Equations: Linear Equations

- The most general first order differential equation can be written as,

$$\frac{dy}{dt} = f(y,t) \qquad \textbf{(1)}$$

- We solve linear first order differential equations, *i.e.* differential equations in the form

$$\frac{dy}{dt} + p(t)y = g(t)$$

> To solve the linear equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $v(x) = e^{\int P(x)\,dx}$ and integrate both sides.

- When you integrate the product on the left hand side in this procedure, you always obtain the product $v(x)y$ of the integrating factor and solution function y because of the way v is defined.

$$v(x)\frac{dy}{dx} + P(x)v(x)y = v(x)Q(x)$$
$$\frac{d}{dx}(v(x)\cdot y) = v(x)Q(x)$$
$$v(x)\cdot y = \int v(x)Q(x)\,dx$$
$$y = \frac{1}{v(x)}\int v(x)Q(x)\,dx$$

$$\frac{d}{dx}(vy) = v\frac{dy}{dx} + Pvy$$
$$v\frac{dy}{dx} + y\frac{dv}{dx} = v\frac{dy}{dx} + Pvy$$
$$y\frac{dv}{dx} = Pvy$$

$$\frac{dv}{dx} = Pv$$
$$\frac{dv}{v} = P\,dx$$
$$\int\frac{dv}{v} = \int P\,dx$$

$$\ln v = \int P\,dx$$
$$e^{\ln v} = e^{\int P\,dx}$$
$$v = e^{\int P\,dx}$$

First Order Differential Equations: Linear Equations

EXAMPLE 2 Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution First we put the equation in standard form (Example 1):

$$\frac{dy}{dx} - \frac{3}{x}y = x,$$

The integrating factor is

$$\begin{aligned} v(x) &= e^{\int P(x)\,dx} = e^{\int (-3/x)\,dx} \\ &= e^{-3\ln|x|} \\ &= e^{-3\ln x} \qquad x > 0 \\ &= e^{\ln x^{-3}} = \frac{1}{x^3}. \end{aligned}$$

$$\frac{1}{x^3}\cdot\left(\frac{dy}{dx} - \frac{3}{x}y\right) = \frac{1}{x^3}\cdot x$$

$$\frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y = \frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{1}{x^3}y\right) = \frac{1}{x^2} \qquad \text{Left-hand side is } \frac{d}{dx}(v\cdot y).$$

$$\frac{1}{x^3}y = \int \frac{1}{x^2}\,dx \qquad \text{Integrate both sides.}$$

$$\frac{1}{x^3}y = -\frac{1}{x} + C.$$

Solving this last equation for y gives the general solution:

$$y = x^3\left(-\frac{1}{x} + C\right) = -x^2 + Cx^3, \qquad x > 0.$$

First Order Differential Equations: Linear Equations

EXAMPLE 3 Find the particular solution of

$$3xy' - y = \ln x + 1, \qquad x > 0,$$

satisfying $y(1) = -2$.

Solution With $x > 0$, we write the equation in standard form:

$$y' - \frac{1}{3x}y = \frac{\ln x + 1}{3x}.$$

Then the integrating factor is given by

$$v = e^{\int -dx/3x} = e^{(-1/3)\ln x} = x^{-1/3}. \qquad x > 0$$

Thus

$$x^{-1/3}y = \frac{1}{3}\int (\ln x + 1)x^{-4/3}\,dx. \qquad \text{Left-hand side is } vy.$$

Integration by parts of the right-hand side gives

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) + \int x^{-4/3}\,dx + C.$$

Therefore

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) - 3x^{-1/3} + C$$

When $x = 1$ and $y = -2$ this last equation becomes $-2 = -(0 + 4) + C$, so $C = 2$.

Substitution into the equation for y gives the particular solution

$$y = 2x^{1/3} - \ln x - 4.$$

First Order Differential Equations: Separable Equations

A separable differential equation is any differential equation that we can write in the following form.

$$N(y)\frac{dy}{dx} = M(x) \qquad \textbf{(1)}$$

Then collect all y terms with dy and all x terms with dx:

$$N(y)\,dy = M(x)\,dx$$

Now we integrate both sides of this to get,

$$\int N(y)\,dy = \int M(x)\,dx$$

EXAMPLE 1 Solve the differential equation

$$\frac{dy}{dx} = (1 + y)e^x, \quad y > -1.$$

Solution Since $1 + y$ is never zero for $y > -1$, we can solve the equation by separating the variables.

$$\int \frac{dy}{1 + y} = \int e^x\,dx \qquad \text{Integrate both sides.}$$

$$\ln(1 + y) = e^x + C \qquad C \text{ represents the combined constants of integration.}$$

The last equation gives y as an implicit function of x.

First Order Differential Equations: Separable Equations

EXAMPLE 2 Solve the equation $y(x+1)\dfrac{dy}{dx} = x(y^2+1)$.

Solution We change to differential form, separate the variables, and integrate:

$$y(x+1)\,dy = x(y^2+1)\,dx$$

$$\frac{y\,dy}{y^2+1} = \frac{x\,dx}{x+1} \qquad x \neq -1$$

$$\int \frac{y\,dy}{1+y^2} = \int \left(1 - \frac{1}{x+1}\right) dx \qquad \text{Divide } x \text{ by } x+1.$$

$$\frac{1}{2}\ln(1+y^2) = x - \ln|x+1| + C.$$

The last equation gives the solution y as an implicit function of x.

First Order Differential Equations: Exact Equations

Suppose that we have the following differential equation.

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

Now, if there is a function $\Psi(x,y)$, so that,

$$\Psi_x = M(x,y) \qquad \text{and} \qquad \Psi_y = N(x,y)$$

then we call the differential equation **exact**. In these cases we can write the differential equation as

$$\Psi_x + \Psi_y \frac{dy}{dx} = 0$$

Then using the chain rule from your multi-variable calculus class we can further reduce the differential equation to the following derivative,

$$\frac{d}{dx}\left(\Psi\left(x, y(x)\right)\right) = 0$$

The (implicit) solution to an exact differential equation is then

$$\Psi(x,y) = c$$

However, we also have the following.

$$\Psi_{xy} = \left(\Psi_x\right)_y = \left(M\right)_y = M_y$$

$$\Psi_{yx} = \left(\Psi_y\right)_x = \left(N\right)_x = N_x$$

Therefore, if a differential equation is exact and $\Psi(x,y)$ meets all of its continuity conditions we must have.

$$M_y = N_x$$

First Order Differential Equations: Exact Equations

Example Solve the following IVP

$$2xy-9x^2+\left(2y+x^2+1\right)\frac{dy}{dx}=0, \qquad y(0)=-3$$

Solution

$$M=2xy-9x^2 \qquad M_y=2x$$

$$N=2y+x^2+1 \qquad N_x=2x$$

So, the differential equation is exact according to the test. Well recall that

$$\Psi_x=M$$
$$\Psi_y=N$$

$$\Psi=\int M\,dx \qquad \text{OR} \qquad \Psi=\int N\,dy$$

So, we'll use the first one.

$$\Psi(x,y)=\int 2xy-9x^2\,dx=x^2y-3x^3+h(y)$$

$$\Psi_y=x^2+h'(y)=2y+x^2+1=N$$

$$h'(y)=2y+1$$

$$h(y)=\int 2y+1\,dy=y^2+y+k$$

So, we can now write down $\Psi(x,y)$. $\Psi(x,y)=x^2y-3x^3+y^2+y=y^2+\left(x^2+1\right)y-3x^3=c$

This is where we left off in the first example. Let's now apply the initial condition to find *c*.

$$(-3)^2+(0+1)(-3)-3(0)^3=c \qquad \Rightarrow \qquad c=6$$

The implicit solution is then.

$$y^2+\left(x^2+1\right)y-3x^3-6=0$$

Bernoulli Differential Equations

Differential equations in the following form are called **Bernoulli Equations**.

$$y' + p(x)y = q(x)y^n$$

In order to solve these we'll first divide the differential equation by y^n to get,

$$y^{-n}y' + p(x)y^{1-n} = q(x)$$

We are now going to use the substitution $v = y^{1-n}$. So, taking the derivative

$$v' = (1-n)y^{-n}y'$$

Now, plugging this as well as our substitution into the differential equation gives,

$$\frac{1}{1-n}v' + p(x)v = q(x)$$

This is a linear differential equation that we can solve for *v* and once we have this in hand we can also get the solution to the original differential equation by plugging *v* back into our substitution and solving for *y*.

Bernoulli Differential Equations

Example 1 Solve the following IVP

$$y' + \frac{4}{x}y = x^3 y^2 \qquad y(2) = -1, \qquad x > 0$$

Solution $y^{-2}y' + \frac{4}{x}y^{-1} = x^3$

The substitution and derivative that we'll need here is,

$$v = y^{-1} \qquad v' = -y^{-2}y'$$

With this substitution the differential equation becomes,

$$-v' + \frac{4}{x}v = x^3$$

Here's the solution to this differential equation.

$$v' - \frac{4}{x}v = -x^3 \qquad \Rightarrow \qquad \mu(x) = \mathrm{e}^{\int -\frac{4}{x}dx} = \mathrm{e}^{-4\ln|x|} = x^{-4}$$

$$\int \left(x^{-4}v\right)' dx = \int -x^{-1}\,dx$$

$$x^{-4}v = -\ln|x| + c \qquad \Rightarrow \qquad v(x) = cx^4 - x^4 \ln x$$

So, to get the solution in terms of *y* all we need to do is plug the substitution back in. Doing this gives,

$$y^{-1} = x^4(c - \ln x)$$

$$(-1)^{-1} = c2^4 - 2^4 \ln 2 \qquad \Rightarrow \qquad c = \ln 2 - \frac{1}{16}$$

Plugging in for *c* and solving for *y* gives,

$$y(x) = \frac{1}{x^4\left(\ln 2 - \frac{1}{16} - \ln x\right)} = \frac{-16}{x^4\left(1 + 16\ln x - 16\ln 2\right)} = \frac{-16}{x^4\left(1 + 16\ln\frac{x}{2}\right)}$$

First Order Differential Equations: Substitutions

The first substitution we'll take a look at will require the differential equation to be in the form,

$$y' = F\left(\frac{y}{x}\right)$$

First order differential equations that can be written in this form are called **homogeneous** differential equations. Note that we will usually have to do some rewriting in order to put the differential equation into the proper form. we will use the following substitution.

$$v(x) = \frac{y}{x}$$

We can then rewrite this as, $y = xv$

$$y' = v + xv'$$

Under this substitution the differential equation is then,

$$v + xv' = F(v)$$

$$xv' = F(v) - v \qquad \Rightarrow \qquad \frac{dv}{F(v) - v} = \frac{dx}{x}$$

As we can see with a small rewrite of the new differential equation we will have a separable differential equation after the substitution.

First Order Differential Equations: Substitutions

Example Solve the following IVP

$$x\,y' = y\left(\ln x - \ln y\right) \qquad y(1) = 4, \qquad x > 0$$

Solution we can rewrite this as,

$$y' = \frac{y}{x}\ln\left(\frac{x}{y}\right) \qquad v + xv' = v\ln\left(\frac{1}{v}\right)$$

Applying the substitution and separating gives,

$$xv' = v\left(\ln\left(\frac{1}{v}\right) - 1\right)$$

$$\frac{dv}{v\left(\ln\left(\frac{1}{v}\right)-1\right)} = \frac{dx}{x}$$

Integrate both sides and do a little rewrite to get, $\ln\left(\ln\left(\frac{1}{v}\right)-1\right) = c - \ln x$

$$\ln\left(\tfrac{1}{v}\right) - 1 = \mathbf{e}^{\ln(x)^{-1}+c} = \mathbf{e}^{c}\mathbf{e}^{\ln(x)^{-1}} = \frac{c}{x} \qquad \ln\left(\tfrac{1}{v}\right) = \frac{c}{x} + 1$$

$$\frac{1}{v} = \mathbf{e}^{\frac{c}{x}+1} \qquad \Rightarrow \qquad v = \mathbf{e}^{-\frac{c}{x}-1}$$

Plugging the substitution back in and solving for *y* gives,

$$\frac{y}{x} = \mathbf{e}^{-\frac{c}{x}-1} \qquad \Rightarrow \qquad y(x) = x\mathbf{e}^{-\frac{c}{x}-1}$$

Applying the initial condition and solving for *c* gives, $4 = \mathbf{e}^{-c-1} \qquad \Rightarrow \qquad c = -(1+\ln 4)$

The solution is then,

$$y(x) = x\mathbf{e}^{\frac{1+\ln 4}{x}-1}$$

First Order Differential Equations: Substitutions

For the next substitution we'll take a look at we'll need the differential equation in the form,

$$y' = G(ax + by)$$

In these cases, we'll use the substitution,

$$v = ax + by \qquad \Rightarrow \qquad v' = a + by'$$

Plugging this into the differential equation gives,

$$\tfrac{1}{b}(v' - a) = G(v)$$

$$v' = a + bG(v) \qquad \Rightarrow \qquad \frac{dv}{a + bG(v)} = dx$$

So, with this substitution we'll be able to rewrite the original differential equation as a new separable differential equation that we can solve.

First Order Differential Equations: Substitutions

Example 3 Solve the following IVP

$$y' - (4x - y + 1)^2 = 0 \qquad y(0) = 2$$

Solution

In this case we'll use the substitution.

$$v = 4x - y \qquad v' = 4 - y'$$

So, plugging this into the differential equation gives,

$$4 - v' - (v+1)^2 = 0$$

$$v' = 4 - (v+1)^2$$

$$\frac{dv}{(v+1)^2 - 4} = -dx$$

$$\int \frac{dv}{v^2 + 2v - 3} = \int \frac{dv}{(v+3)(v-1)} = \int -dx$$

$$\frac{1}{4} \int \frac{1}{v-1} - \frac{1}{v+3}\, dv = \int -dx$$

$$\tfrac{1}{4}\left(\ln(v-1) - \ln(v+3)\right) = -x + c$$

$$\ln\left(\frac{v-1}{v+3}\right) = c - 4x$$

$$\frac{v-1}{v+3} = \mathbf{e}^{c-4x} = c\,\mathbf{e}^{-4x}$$

$$v - 1 = c\,\mathbf{e}^{-4x}(v+3)$$

$$v\left(1 - c\,\mathbf{e}^{-4x}\right) = 1 + 3c\,\mathbf{e}^{-4x}$$

So, let's solve for *v* and then go ahead and go back into terms of *y*.

$$y(x) = 4x - \frac{1 + 3c\,\mathbf{e}^{-4x}}{1 - c\,\mathbf{e}^{-4x}}$$

The last step is to then apply the initial condition and solve for *c*.

$$2 = y(0) = -\frac{1+3c}{1-c} \qquad \Rightarrow \qquad c = -3$$

The solution is then,

$$y(x) = 4x - \frac{1 - 9\,\mathbf{e}^{-4x}}{1 + 3\,\mathbf{e}^{-4x}}$$

Second Order Differential Equations

- The most general linear second order differential equation is in the form.

$$p(t)y'' + q(t)y' + r(t)y = g(t) \qquad \textbf{(1)}$$

- The most general linear second order differential equation is in the form.

$$ay'' + by' + cy = g(t) \qquad \textbf{(2)}$$

- When g(t) = 0 we call the differential equation **homogeneous** and when g (t) ≠ 0 we call the differential equation **nonhomogeneous**.

- Here is the general constant coefficient, homogeneous, linear, second order differential equation.

$$ay'' + by' + cy = 0$$

Principle of Superposition

If $y_1(t)$ and $y_2(t)$ are two solutions to a linear, homogeneous differential equation then so is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \qquad \textbf{(3)}$$

Constant-Coefficient Homogeneous Equations

Suppose we wish to solve the second-order homogeneous differential equation

$$ay'' + by' + cy = 0,$$

If we substitute $y = e^{rx}$ into Equation , we obtain

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0.$$

Thus, $y = e^{rx}$ is a solution if and only if r is a solution to the algebraic equation

$$ar^2 + br + c = 0.$$

called the **auxiliary equation** (or **characteristic equation**)

The auxiliary equation is a quadratic equation with roots

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Case 1: $b^2 - 4ac > 0$. In this case the auxiliary equation has two real and unequal roots r_1 and r_2.

Case 2: $b^2 - 4ac = 0$. In this case $r_1 = r_2 = -b/2a$.

Case 3: $b^2 - 4ac < 0$. In this case the auxiliary equation has two complex roots $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$, where α and β are real numbers and $i^2 = -1$. (These real numbers are $\alpha = -b/2a$ and $\beta = \sqrt{4ac - b^2}/2a$.)

Constant-Coefficient Homogeneous Equations

THEOREM 3 If r_1 and r_2 are two real and unequal roots to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is the general solution to $ay'' + by' + cy = 0$.

EXAMPLE 1 Find the general solution of the differential equation

$$y'' - y' - 6y = 0.$$

Solution Substitution of $y = e^{rx}$ into the differential equation yields the auxiliary equation

$$r^2 - r - 6 = 0,$$

which factors as

$$(r - 3)(r + 2) = 0.$$

The roots are $r_1 = 3$ and $r_2 = -2$. Thus, the general solution is

$$y = c_1 e^{3x} + c_2 e^{-2x}.$$ ■

Constant-Coefficient Homogeneous Equations

THEOREM 4 If r is the only (repeated) real root to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = c_1 e^{rx} + c_2 x e^{rx}$$

is the general solution to $ay'' + by' + cy = 0$.

EXAMPLE 2 Find the general solution to

$$y'' + 4y' + 4y = 0.$$

Solution The auxiliary equation is

$$r^2 + 4r + 4 = 0,$$

which factors into

$$(r + 2)^2 = 0.$$

Thus, $r = -2$ is a double root. Therefore, the general solution is

$$y = c_1 e^{-2x} + c_2 x e^{-2x}.$$

■

Constant-Coefficient Homogeneous Equations

THEOREM 5 If $r_1 = \alpha + i\beta$ and $r_2 = \alpha - i\beta$ are two complex roots to the auxiliary equation $ar^2 + br + c = 0$, then

$$y = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x)$$

is the general solution to $ay'' + by' + cy = 0$.

EXAMPLE 3 Find the general solution to the differential equation

$$y'' - 4y' + 5y = 0.$$

Solution The auxiliary equation is

$$r^2 - 4r + 5 = 0.$$

The roots are the complex pair $r = (4 \pm \sqrt{16 - 20})/2$ or $r_1 = 2 + i$ and $r_2 = 2 - i$. Thus, $\alpha = 2$ and $\beta = 1$ give the general solution

$$y = e^{2x}(c_1 \cos x + c_2 \sin x).$$

■

Constant-Coefficient Homogeneous Equations

EXAMPLE 4 Find the particular solution to the initial value problem

$$y'' - 2y' + y = 0, \qquad y(0) = 1, \quad y'(0) = -1.$$

Solution The auxiliary equation is

$$r^2 - 2r + 1 = (r - 1)^2 = 0.$$

The repeated real root is $r = 1$, giving the general solution

$$y = c_1 e^x + c_2 x e^x.$$

Then,

$$y' = c_1 e^x + c_2 (x + 1) e^x.$$

From the initial conditions we have

$$1 = c_1 + c_2 \cdot 0 \qquad \text{and} \qquad -1 = c_1 + c_2 \cdot 1.$$

Thus, $c_1 = 1$ and $c_2 = -2$. The unique solution satisfying the initial conditions is

$$y = e^x - 2xe^x.$$

Nonhomogeneous Linear Equations

In this section we learn how to solve second-order nonhomogeneous linear differential equations with constant coefficients, that is, equations of the form

$$ay'' + by' + cy = G(x) \tag{1}$$

where a, b, and c are constants and G is a continuous function. The related homogeneous equation

$$ay'' + by' + cy = 0 \tag{2}$$

is called the **complementary equation** and plays an important role in the solution of the original nonhomogeneous equation (1).

3 Theorem The general solution of the nonhomogeneous differential equation (1) can be written as

$$y(x) = y_p(x) + y_c(x)$$

where y_p is a particular solution of Equation 1 and y_c is the general solution of the complementary Equation 2.

The method of undetermined coefficients

Summary of the Method of Undetermined Coefficients

1. If $G(x) = e^{kx}P(x)$, where P is a polynomial of degree n, then try $y_p(x) = e^{kx}Q(x)$, where $Q(x)$ is an nth-degree polynomial (whose coefficients are determined by substituting in the differential equation).

2. If $G(x) = e^{kx}P(x)\cos mx$ or $G(x) = e^{kx}P(x)\sin mx$, where P is an nth-degree polynomial, then try

$$y_p(x) = e^{kx}Q(x)\cos mx + e^{kx}R(x)\sin mx$$

where Q and R are nth-degree polynomials.

Modification: If any term of y_p is a solution of the complementary equation, multiply y_p by x (or by x^2 if necessary).

The method of undetermined coefficients

EXAMPLE 2 Solve $y'' + 4y = e^{3x}$.

SOLUTION The auxiliary equation is $r^2 + 4 = 0$ with roots $\pm 2i$, so the solution of the complementary equation is

$$y_c(x) = c_1 \cos 2x + c_2 \sin 2x$$

For a particular solution we try $y_p(x) = Ae^{3x}$. Then $y_p' = 3Ae^{3x}$ and $y_p'' = 9Ae^{3x}$. Substituting into the differential equation, we have

$$9Ae^{3x} + 4(Ae^{3x}) = e^{3x}$$

so $13Ae^{3x} = e^{3x}$ and $A = \frac{1}{13}$. Thus a particular solution is

$$y_p(x) = \frac{1}{13}e^{3x}$$

and the general solution is

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{13}e^{3x}$$

■

The method of undetermined coefficients

EXAMPLE 4 Solve $y'' - 4y = xe^x + \cos 2x$.

SOLUTION The auxiliary equation is $r^2 - 4 = 0$ with roots ± 2, so the solution of the complementary equation is $y_c(x) = c_1 e^{2x} + c_2 e^{-2x}$. For the equation $y'' - 4y = xe^x$ we try

$$y_{p_1}(x) = (Ax + B)e^x$$

Then $y'_{p_1} = (Ax + A + B)e^x$, $y''_{p_1} = (Ax + 2A + B)e^x$, so substitution in the equation gives

$$(Ax + 2A + B)e^x - 4(Ax + B)e^x = xe^x$$

or

$$(-3Ax + 2A - 3B)e^x = xe^x$$

Thus $-3A = 1$ and $2A - 3B = 0$, so $A = -\frac{1}{3}$, $B = -\frac{2}{9}$, and

$$y_{p_1}(x) = \left(-\tfrac{1}{3}x - \tfrac{2}{9}\right)e^x$$

The method of undetermined coefficients

For the equation $y'' - 4y = \cos 2x$, we try

$$y_{p_2}(x) = C\cos 2x + D\sin 2x$$

Substitution gives

$$-4C\cos 2x - 4D\sin 2x - 4(C\cos 2x + D\sin 2x) = \cos 2x$$

or

$$-8C\cos 2x - 8D\sin 2x = \cos 2x$$

Therefore $-8C = 1$, $-8D = 0$, and

$$y_{p_2}(x) = -\tfrac{1}{8}\cos 2x$$

By the superposition principle, the general solution is

$$y = y_c + y_{p_1} + y_{p_2} = c_1 e^{2x} + c_2 e^{-2x} - \left(\tfrac{1}{3}x + \tfrac{2}{9}\right)e^x - \tfrac{1}{8}\cos 2x \quad \blacksquare$$

The method of undetermined coefficients

For the equation $y'' - 4y = \cos 2x$, we try

$$y_{p_2}(x) = C \cos 2x + D \sin 2x$$

Substitution gives

$$-4C \cos 2x - 4D \sin 2x - 4(C \cos 2x + D \sin 2x) = \cos 2x$$

or

$$-8C \cos 2x - 8D \sin 2x = \cos 2x$$

Therefore $-8C = 1$, $-8D = 0$, and

$$y_{p_2}(x) = -\tfrac{1}{8} \cos 2x$$

By the superposition principle, the general solution is

$$y = y_c + y_{p_1} + y_{p_2} = c_1 e^{2x} + c_2 e^{-2x} - \left(\tfrac{1}{3}x + \tfrac{2}{9}\right)e^x - \tfrac{1}{8} \cos 2x \quad \blacksquare$$

The method of undetermined coefficients

EXAMPLE 5 Solve $y'' + y = \sin x$.

SOLUTION The auxiliary equation is $r^2 + 1 = 0$ with roots $\pm i$, so the solution of the complementary equation is

$$y_c(x) = c_1 \cos x + c_2 \sin x$$

$$y_p(x) = Ax \cos x + Bx \sin x$$

Then $$y_p'(x) = A \cos x - Ax \sin x + B \sin x + Bx \cos x$$

$$y_p''(x) = -2A \sin x - Ax \cos x + 2B \cos x - Bx \sin x$$

Substitution in the differential equation gives

$$y_p'' + y_p = -2A \sin x + 2B \cos x = \sin x$$

so $A = -\frac{1}{2}$, $B = 0$, and $y_p(x) = -\frac{1}{2}x \cos x$

The general solution is

$$y(x) = c_1 \cos x + c_2 \sin x - \tfrac{1}{2}x \cos x$$

The method of undetermined coefficients

EXAMPLE 6 Determine the form of the trial solution for the differential equation $y'' - 4y' + 13y = e^{2x}\cos 3x$.

SOLUTION Here $G(x)$ has the form of part 2 of the summary, where $k = 2$, $m = 3$, and $P(x) = 1$. So, at first glance, the form of the trial solution would be

$$y_p(x) = e^{2x}(A\cos 3x + B\sin 3x)$$

But the auxiliary equation is $r^2 - 4r + 13 = 0$, with roots $r = 2 \pm 3i$, so the solution of the complementary equation is

$$y_c(x) = e^{2x}(c_1\cos 3x + c_2\sin 3x)$$

This means that we have to multiply the suggested trial solution by x. So, instead, we use

$$y_p(x) = xe^{2x}(A\cos 3x + B\sin 3x)$$

■

Thank you for listening.

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