Lecture 2

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Laplace Transforms

Suppose that *f(t)* is a piecewise continuous function. The Laplace transform of *f(t)* is denoted $\mathcal{L}\{f(t)\}$ and defined as

$$\mathcal{L}\{f(t)\} = \int_0^\infty \mathbf{e}^{-st} f(t)\, dt \qquad \textbf{(1)}$$

$$\mathcal{L}\{f(t)\} = F(s)$$

Example 2 Compute $\mathcal{L}\{1\}$.

Solution

$$\mathcal{L}\{1\} = \int_0^\infty \mathbf{e}^{-st}\, dt = -\frac{1}{-s} \qquad \text{provided } -s < 0$$

$$\mathcal{L}\{1\} = \frac{1}{s} \qquad \text{provided } s > 0$$

Example 3 Compute $\mathcal{L}\{\mathbf{e}^{at}\}$

Solution

$$\mathcal{L}\{\mathbf{e}^{at}\} = \int_0^\infty \mathbf{e}^{-st}\mathbf{e}^{at}\, dt = \int_0^\infty \mathbf{e}^{(a-s)t}\, dt$$

$$= -\frac{1}{a-s} \qquad \text{provided } a - s < 0$$

$$= \frac{1}{s-a} \qquad \text{provided } s > a$$

Laplace Transforms

Example 4

$$\mathcal{L}\{\sin(at)\} = F(s)$$

$$= \int_0^{\infty} \mathbf{e}^{-st} \sin(at)\,dt$$

$$= \lim_{n\to\infty} \int_0^{n} \mathbf{e}^{-st} \sin(at)\,dt$$

Now, if we integrate by parts we will arrive at,

$$F(s) = \lim_{n\to\infty}\left(-\left(\frac{1}{a}\mathbf{e}^{-st}\cos(at)\right)\Bigg|_0^n - \frac{s}{a}\int_0^n \mathbf{e}^{-st}\cos(at)\,dt \right)$$

$$\Downarrow$$

$$\mathcal{L}\{\sin(at)\} = F(s) = \frac{a}{s^2+a^2} \qquad \text{provided } s > 0$$

Table Of Laplace Transforms

	$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$
2.	$\mathbf{e}^{at}$	$\frac{1}{s-a}$
3.	$t^n, \quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$
8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$

Table Of Laplace Transforms

17.	$\sinh(at)$	$\dfrac{a}{s^2-a^2}$
18.	$\cosh(at)$	$\dfrac{s}{s^2-a^2}$
19.	$\mathbf{e}^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
20.	$\mathbf{e}^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
21.	$\mathbf{e}^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$
22.	$\mathbf{e}^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
23.	$t^n\mathbf{e}^{at},\quad n=1,2,3,\ldots$	$\dfrac{n!}{(s-a)^{n+1}}$
24.	$f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right)$

Table Of Laplace Transforms

29.	$\mathbf{e}^{ct} f(t)$	$F(s-c)$
30.	$t^n f(t), \quad n = 1, 2, 3, \ldots$	$(-1)^n F^{(n)}(s)$
35.	$f'(t)$	$sF(s) - f(0)$
36.	$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
37.	$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$

Fact

Given *f(t)* and *g(t)* then,

$$\mathcal{L}\{af(t) + bg(t)\} = aF(s) + bG(s)$$

for any constants *a* and *b*.

Laplace Transforms

Example 1 Find the Laplace transforms of the given functions.

(a) $f(t) = 6e^{-5t} + e^{3t} + 5t^3 - 9$

$$F(s) = 6\frac{1}{s-(-5)} + \frac{1}{s-3} + 5\frac{3!}{s^{3+1}} - 9\frac{1}{s}$$

$$= \frac{6}{s+5} + \frac{1}{s-3} + \frac{30}{s^4} - \frac{9}{s}$$

(b) $g(t) = 4\cos(4t) - 9\sin(4t) + 2\cos(10t)$

$$G(s) = 4\frac{s}{s^2+(4)^2} - 9\frac{4}{s^2+(4)^2} + 2\frac{s}{s^2+(10)^2}$$

$$= \frac{4s}{s^2+16} - \frac{36}{s^2+16} + \frac{2s}{s^2+100}$$

(c) $h(t) = 3\sinh(2t) + 3\sin(2t)$

$$H(s) = 3\frac{2}{s^2-(2)^2} + 3\frac{2}{s^2+(2)^2}$$

$$= \frac{6}{s^2-4} + \frac{6}{s^2+4}$$

Laplace Transforms

Example 1 Find the Laplace transforms of the given functions.

(d) $g(t) = e^{3t} + \cos(6t) - e^{3t}\cos(6t)$

$$G(s) = \frac{1}{s-3} + \frac{s}{s^2+(6)^2} - \frac{s-3}{(s-3)^2+(6)^2}$$

$$= \frac{1}{s-3} + \frac{s}{s^2+36} - \frac{s-3}{(s-3)^2+36}$$

(e) $f(t) = tg'(t)$

This final part will again use #30 from the table as well as #35.

$$\mathcal{L}\{tg'(t)\} = -\frac{d}{ds}\mathcal{L}\{g'\}$$

$$= -\frac{d}{ds}\{sG(s) - g(0)\}$$

$$= -(G(s) + sG'(s) - 0)$$

$$= -G(s) - sG'(s)$$

Laplace Transforms

Example 2 Find the transform of each of the following functions.

(a) $f(t) = t\cosh(3t)$

This will correspond to #30 if we take *n=1*.

$$F(s) = \mathcal{L}\{tg(t)\} = -G'(s), \quad \text{where } g(t) = \cosh(3t)$$

So, we then have,

$$G(s) = \frac{s}{s^2 - 9} \qquad G'(s) = -\frac{s^2 + 9}{(s^2 - 9)^2}$$

Using #30 we then have,

$$F(s) = \frac{s^2 + 9}{(s^2 - 9)^2}$$

(b) $h(t) = t^2 \sin(2t)$

$$H(s) = \mathcal{L}\{tf(t)\} = -F'(s), \qquad \text{where } f(t) = t\sin(2t)$$

So, using #9 we have,

$$F(s) = \frac{4s}{(s^2 + 4)^2} \qquad F'(s) = -\frac{12s^2 - 16}{(s^2 + 4)^3}$$

The transform is then,

$$H(s) = \frac{12s^2 - 16}{(s^2 + 4)^3}$$

Inverse Laplace Transforms

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

Fact

Given the two Laplace transforms *F(s)* and *G(s)* then

$$\mathcal{L}^{-1}\{aF(s) + bG(s)\} = a\mathcal{L}^{-1}\{F(s)\} + b\mathcal{L}^{-1}\{G(s)\}$$

for any constants *a* and *b*.

Example 1 Find the inverse transform of each of the following.

(a) $F(s) = \frac{6}{s} - \frac{1}{s-8} + \frac{4}{s-3}$

$$F(s) = 6\frac{1}{s} - \frac{1}{s-8} + 4\frac{1}{s-3}$$

$$f(t) = 6(1) - \mathbf{e}^{8t} + 4\left(\mathbf{e}^{3t}\right)$$

$$= 6 - \mathbf{e}^{8t} + 4\mathbf{e}^{3t}$$

(b) $H(s) = \frac{19}{s+2} - \frac{1}{3s-5} + \frac{7}{s^5}$

$$= 19\frac{1}{s-(-2)} - \frac{1}{3}\frac{1}{s-\frac{5}{3}} + \frac{7}{4!}\frac{4!}{s^{4+1}}$$

Let's now take the inverse transform.

$$h(t) = 19\mathbf{e}^{-2t} - \frac{1}{3}\mathbf{e}^{\frac{5t}{3}} + \frac{7}{24}t^4$$

Inverse Laplace Transforms

Example 1 Find the inverse transform of each of the following.

(c) $F(s) = \frac{6s}{s^2+25} + \frac{3}{s^2+25}$

The transform becomes,

$$F(s) = 6\frac{s}{s^2+(5)^2} + \frac{3}{5}\frac{5}{s^2+(5)^2}$$

Taking the inverse transform gives,

$$f(t) = 6\cos(5t) + \frac{3}{5}\sin(5t)$$

(d) $G(s) = \frac{8}{3s^2+12} + \frac{3}{s^2-49}$

$$= \frac{1}{3}\frac{(4)(2)}{s^2+(2)^2} + \frac{3\frac{7}{7}}{s^2-(7)^2}$$

The inverse transform is then,

$$g(t) = \frac{4}{3}\sin(2t) + \frac{3}{7}\sinh(7t)$$

Inverse Laplace Transforms

Example 2 Find the inverse transform of each of the following.

(a) $F(s)=\dfrac{6s-5}{s^2+7}$

$$F(s)=\frac{6s}{s^2+7}-\frac{5\frac{\sqrt{7}}{\sqrt{7}}}{s^2+7}$$

$$f(t)=6\cos\left(\sqrt{7}t\right)-\frac{5}{\sqrt{7}}\sin\left(\sqrt{7}t\right)$$

(b) $F(s)=\dfrac{1-3s}{s^2+8s+21}$

$$s^2+8s+21=s^2+8s+16-16+21$$
$$=s^2+8s+16+5$$
$$=(s+4)^2+5$$

$$F(s)=\frac{-3(s+4)+13}{(s+4)^2+5}$$

$$F(s)=-3\frac{s+4}{(s+4)^2+5}+\frac{13\frac{\sqrt{5}}{\sqrt{5}}}{(s+4)^2+5}$$

$$f(t)=-3\mathbf{e}^{-4t}\cos\left(\sqrt{5}t\right)+\frac{13}{\sqrt{5}}\mathbf{e}^{-4t}\sin\left(\sqrt{5}t\right)$$

Inverse Laplace Transforms

Example 2 Find the inverse transform of each of the following.

(c) $G(s)=\dfrac{3s-2}{2s^2-6s-2}$

$$=\frac{1}{2}\frac{3s-2}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}} \qquad =\frac{1}{2}\frac{3\left(s-\frac{3}{2}\right)+\frac{5}{2}}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}$$

$$=\frac{1}{2}\left(\frac{3\left(s-\frac{3}{2}\right)}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}+\frac{\frac{5}{2}\frac{\sqrt{13}}{\sqrt{13}}}{\left(s-\frac{3}{2}\right)^2-\frac{13}{4}}\right)$$

$$g(t)=\frac{1}{2}\left(3\mathbf{e}^{\frac{3t}{2}}\cosh\left(\frac{\sqrt{13}}{2}t\right)+\frac{5}{\sqrt{13}}\mathbf{e}^{\frac{3t}{2}}\sinh\left(\frac{\sqrt{13}}{2}t\right)\right)$$

(d) $H(s)=\dfrac{s+7}{s^2-3s-10}$

$$H(s)=\frac{s+7}{(s+2)(s-5)} \qquad =\frac{A}{s+2}+\frac{B}{s-5}$$

$$H(s)=\frac{-\frac{5}{7}}{s+2}+\frac{\frac{12}{7}}{s-5}$$

We can now easily do the inverse transform to get,

$$h(t)=-\frac{5}{7}\mathbf{e}^{-2t}+\frac{12}{7}\mathbf{e}^{5t}$$

Inverse Laplace Transforms

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

Example 3 Find the inverse transform of each of the following.

(a) $G(s)=\dfrac{86s-78}{(s+3)(s-4)(5s-1)}$

(b) $F(s)=\dfrac{2-5s}{(s-6)(s^2+11)}$

(c) $G(s)=\dfrac{25}{s^3(s^2+4s+5)}$

Solving IVP's with Laplace Transforms

Recall that the Laplace transform of the first two derivatives:

$$\mathcal{L}\{y'\} = sY(s) - y(0)$$

$$\mathcal{L}\{y''\} = s^2Y(s) - sy(0) - y'(0)$$

Example 1 Solve the following IVP.

$$y'' - 10y' + 9y = 5t,$$

$$y(0) = -1 \quad y'(0) = 2$$

Solution

$$\mathcal{L}\{y''\} - 10\mathcal{L}\{y'\} + 9\mathcal{L}\{y\} = \mathcal{L}\{5t\}$$

$$s^2Y(s) - sy(0) - y'(0) - 10(sY(s) - y(0)) + 9Y(s) = \frac{5}{s^2}$$

$$(s^2 - 10s + 9)Y(s) + s - 12 = \frac{5}{s^2}$$

$$Y(s) = \frac{5}{s^2(s-9)(s-1)} + \frac{12-s}{(s-9)(s-1)}$$

$$Y(s) = \frac{5 + 12s^2 - s^3}{s^2(s-9)(s-1)}$$

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-9} + \frac{D}{s-1}$$

$$5 + 12s^2 - s^3 = As(s-9)(s-1) + B(s-9)(s-1) + Cs^2(s-1) + Ds^2(s-9)$$

$s = 0$	$5 = 9B$	$\Rightarrow$	$B = \frac{5}{9}$
$s = 1$	$16 = -8D$	$\Rightarrow$	$D = -2$
$s = 9$	$248 = 648C$	$\Rightarrow$	$C = \frac{31}{81}$
$s = 2$	$45 = -14A + \frac{4345}{81}$	$\Rightarrow$	$A = \frac{50}{81}$

$$Y(s) = \frac{\frac{50}{81}}{s} + \frac{\frac{5}{9}}{s^2} + \frac{\frac{31}{81}}{s-9} - \frac{2}{s-1}$$

$$y(t) = \frac{50}{81} + \frac{5}{9}t + \frac{31}{81}\mathbf{e}^{9t} - 2\mathbf{e}^{t}$$

Solving IVP's with Laplace Transforms

Example 2 Solve the following IVP.

$$2y'' + 3y' - 2y = te^{-2t}, \qquad y(0) = 0 \quad y'(0) = -2$$

Solution

$$2\left(s^2Y(s) - sy(0) - y'(0)\right) + 3\left(sY(s) - y(0)\right) - 2Y(s) = \frac{1}{(s+2)^2}$$

$$\left(2s^2 + 3s - 2\right)Y(s) + 4 = \frac{1}{(s+2)^2}$$

$$Y(s) = \frac{1}{(2s-1)(s+2)^3} - \frac{4}{(2s-1)(s+2)} = \frac{-4s^2 - 16s - 15}{(2s-1)(s+2)^3}$$

The partial fraction decomposition is then,

$$Y(s) = \frac{A}{2s-1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} + \frac{D}{(s+2)^3}$$

$$\Rightarrow \quad A = -\frac{192}{125} \quad B = \frac{96}{125} \quad C = -\frac{2}{25} \quad D = -\frac{1}{5}$$

$$Y(s) = \frac{1}{125}\left(\frac{-192}{2\left(s - \frac{1}{2}\right)} + \frac{96}{s+2} - \frac{10}{(s+2)^2} - \frac{25\frac{2!}{2!}}{(s+2)^3}\right)$$

Taking the inverse transform then gives,

$$y(t) = \frac{1}{125}\left(-96e^{\frac{t}{2}} + 96e^{-2t} - 10te^{-2t} - \frac{25}{2}t^2e^{-2t}\right)$$

Solving IVP's with Laplace Transforms

Example 3 Solve the following IVP.

$$y'' - 6y' + 15y = 2\sin(3t), \qquad y(0) = -1 \quad y'(0) = -4$$

Solution

$$s^2Y(s) - sy(0) - y'(0) - 6(sY(s) - y(0)) + 15Y(s) = 2\frac{3}{s^2+9}$$

$$Y(s) = \frac{-s^3 + 2s^2 - 9s + 24}{(s^2+9)(s^2-6s+15)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2-6s+15} \quad \Rightarrow \quad A = \frac{1}{10} \quad B = \frac{1}{10} \quad C = -\frac{11}{10} \quad D = \frac{5}{2}$$

$$Y(s) = \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11s+25}{s^2-6s+15}\right)$$

$$= \frac{1}{10}\left(\frac{s+1}{s^2+9} + \frac{-11(s-3+3)+25}{(s-3)^2+6}\right)$$

$$= \frac{1}{10}\left(\frac{s}{s^2+9} + \frac{1\frac{3}{3}}{s^2+9} - \frac{11(s-3)}{(s-3)^2+6} - \frac{8\frac{\sqrt{6}}{\sqrt{6}}}{(s-3)^2+6}\right)$$

Finally, take the inverse transform.

$$y(t) = \frac{1}{10}\left(\cos(3t) + \frac{1}{3}\sin(3t) - 11\mathbf{e}^{3t}\cos\left(\sqrt{6}t\right) - \frac{8}{\sqrt{6}}\mathbf{e}^{3t}\sin\left(\sqrt{6}t\right)\right)$$

Nonconstant Coefficient IVP's

Example 1 Solve the following IVP.

$$y'' + 3ty' - 6y = 2. \qquad y(0) = 0 \quad y'(0) = 0$$

Solution

$$\mathcal{L}\{ty'\} = -\frac{d}{ds}\left(\mathcal{L}\{y'\}\right) = -\frac{d}{ds}\left(sY(s) - y(0)\right) = -sY'(s) - Y(s)$$

$$s^2Y(s) - sy(0) - y'(0) + 3\left(-sY'(s) - Y(s)\right) - 6Y(s) = \frac{2}{s}$$

$$-3sY'(s) + \left(s^2 - 9\right)Y(s) = \frac{2}{s}$$

$$Y'(s) + \left(\frac{3}{s} - \frac{s}{3}\right)Y(s) = -\frac{2}{3s^2}$$

The integrating factor for this differential equation is,

$$\mu(s) = \mathbf{e}^{\int\left(\frac{3}{s} - \frac{s}{3}\right)ds} = \mathbf{e}^{\ln\left(s^3\right) - \frac{s^2}{6}} = s^3\mathbf{e}^{-\frac{s^2}{6}}$$

$$\int\left(s^3\mathbf{e}^{-\frac{s^2}{6}}Y(s)\right)' ds = \int -\frac{2}{3}s\mathbf{e}^{-\frac{s^2}{6}}\, ds$$

$$s^3\mathbf{e}^{-\frac{s^2}{6}}Y(s) = 2\mathbf{e}^{-\frac{s^2}{6}} + c$$

$$Y(s) = \frac{2}{s^3} + c\frac{\mathbf{e}^{\frac{s^2}{6}}}{s^3}$$

$$\lim_{s\to\infty}\left(\frac{2}{s^3} + \frac{c\mathbf{e}^{\frac{s^2}{6}}}{s^3}\right) = 0$$

The second term however, will only go to zero if $c = 0$.

So, the transform of our solution, as well as the solution is,

$$Y(s) = \frac{2}{s^3} \qquad y(t) = t^2$$

Nonconstant Coefficient IVP's

Example 2 Solve the following IVP.

$$ty'' - ty' + y = 2, \qquad y(0) = 2 \quad y'(0) = -4$$

Solution

$$\mathcal{L}\{ty'\} = -sY'(s) - Y(s)$$

$$\mathcal{L}\{ty''\} = -\frac{d}{ds}\left(\mathcal{L}\{y''\}\right) = -\frac{d}{ds}\left(s^2Y(s) - sy(0) - y'(0)\right) = -s^2Y'(s) - 2sY(s) + y(0)$$

Taking the Laplace transform of everything and plugging in the initial conditions gives,

$$Y'(s) + \frac{2}{s}Y(s) = \frac{2}{s^2}$$

Upon solving the differential equation we get,

$$Y(s) = \frac{2}{s} + \frac{c}{s^2}$$

Taking the inverse transform gives,

$$y(t) = 2 + ct$$

Upon differentiating and plugging in the second initial condition we can see that $c = -4$.

So, the solution to this IVP is,

$$y(t) = 2 - 4t$$

Thank you for listening.

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