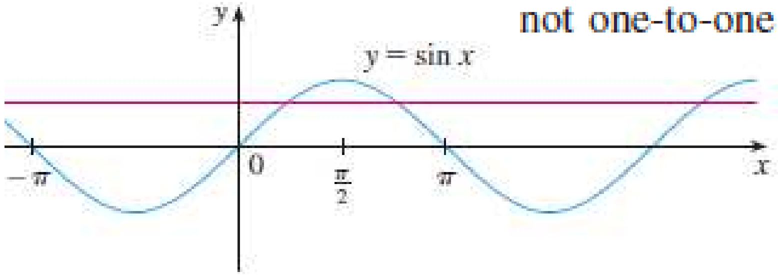
Lecture 1

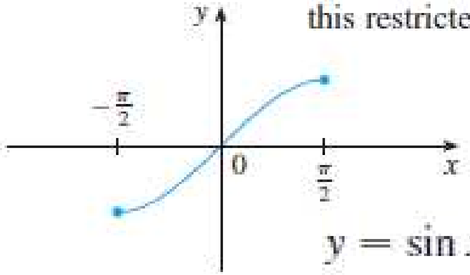
Dr. Moataz El-Zekey

Inverse Trigonometric Functions



not one-to-one

this restricted sine function is one-to-one



$y = \sin x,\ -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$

Since the definition of an inverse function says that

$$f^{-1}(x) = y \iff f(y) = x$$

we have

$$\sin^{-1}x = y \iff \sin y = x \quad \text{and} \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

Thus, if $-1 \leqslant x \leqslant 1$, $\sin^{-1}x$ is the number between $-\pi/2$ and $\pi/2$ whose sine is x.

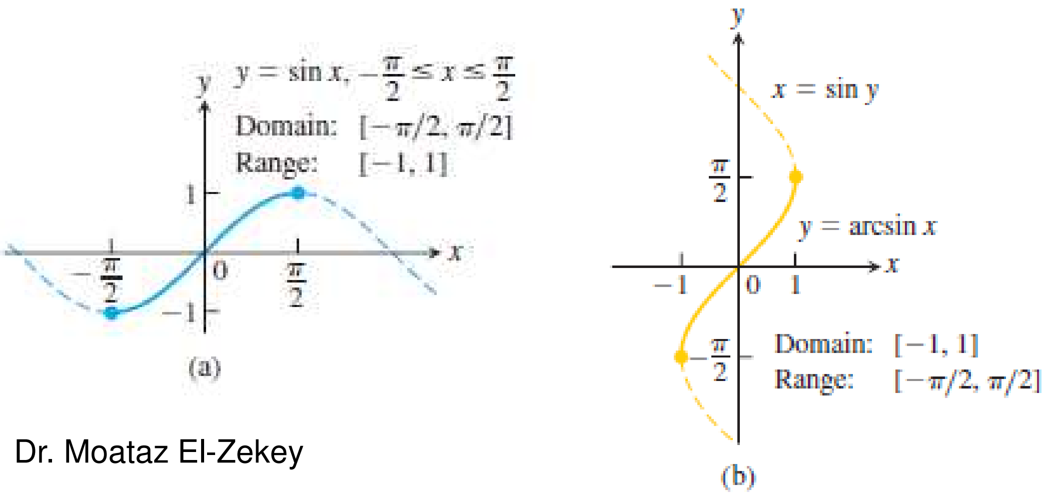
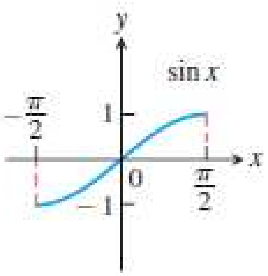


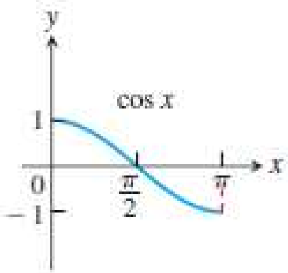
FIGURE 1.63 The graphs of (a) $y = \sin x, -\pi/2 \leq x \leq \pi/2$, and (b) its inverse, $y = \arcsin x$. The graph of $\arcsin x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

Inverse Trigonometric Functions

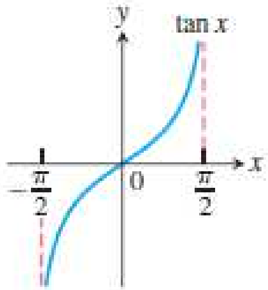
Domain restrictions that make the trigonometric functions one-to-one



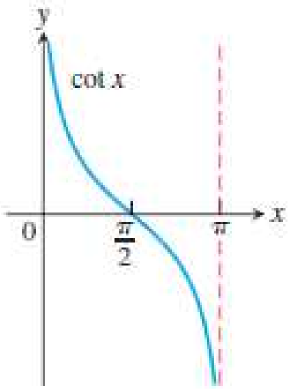
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



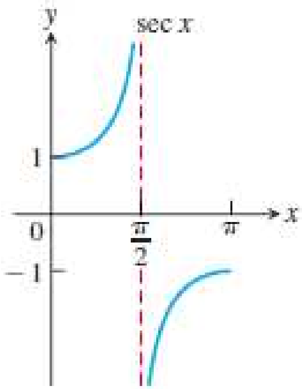
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



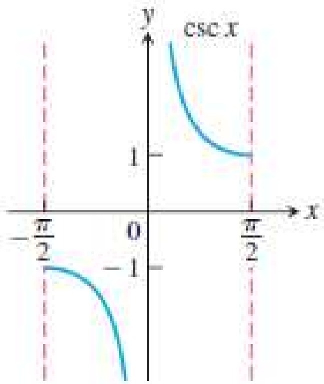
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$



$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Inverse Trigonometric Functions

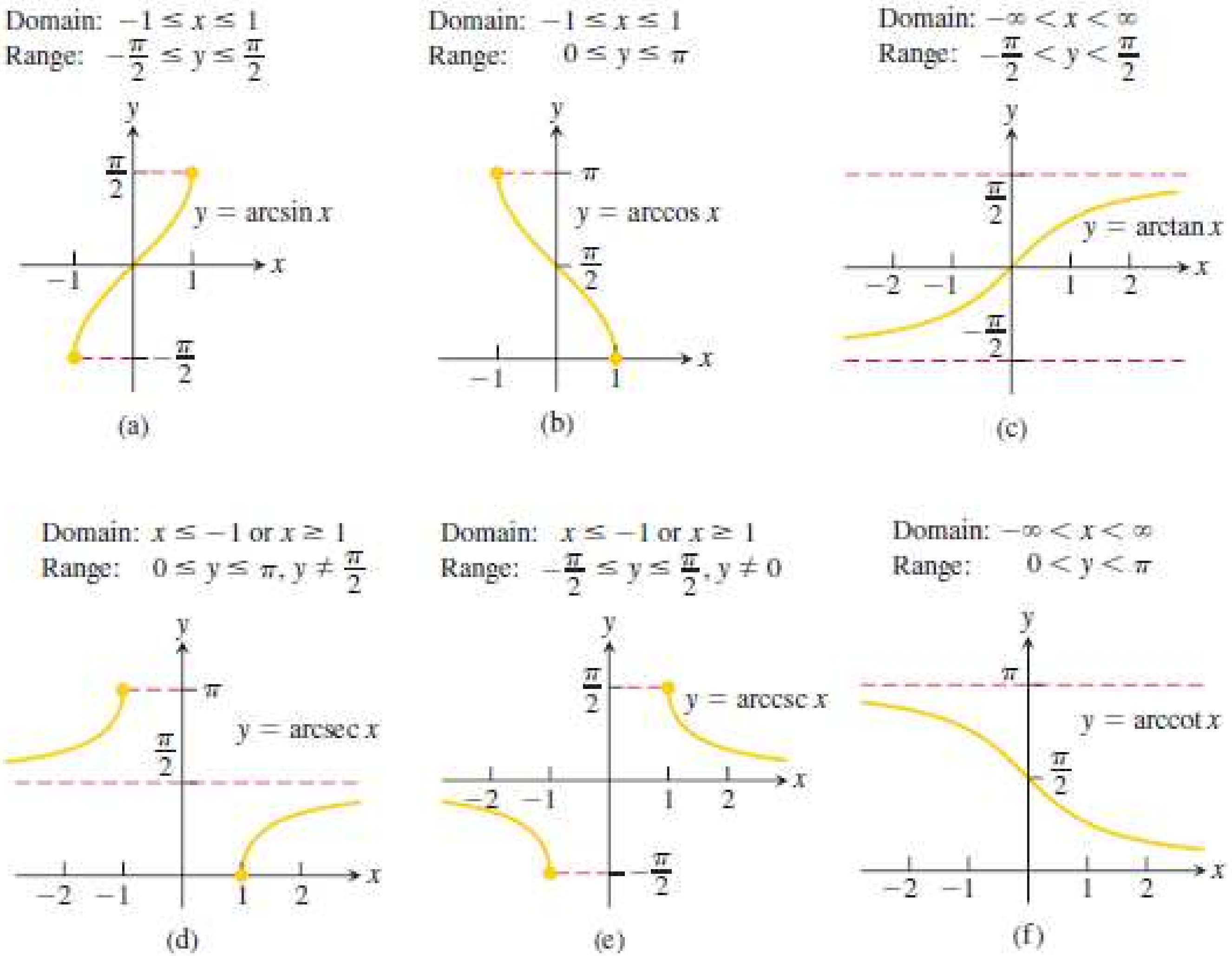


FIGURE 1.64 Graphs of the six basic inverse trigonometric functions.

Inverse Trigonometric Functions

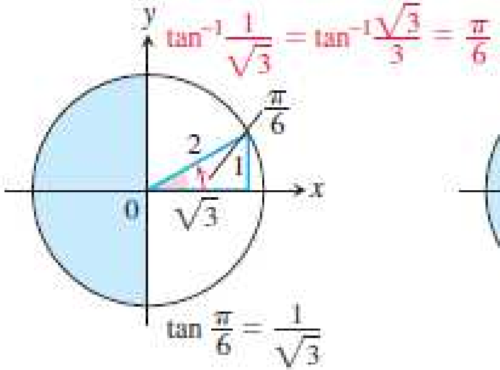
DEFINITIONS

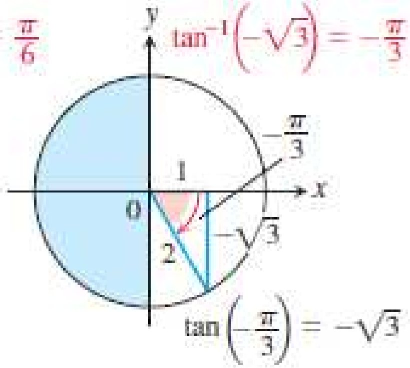
$y = \tan^{-1} x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

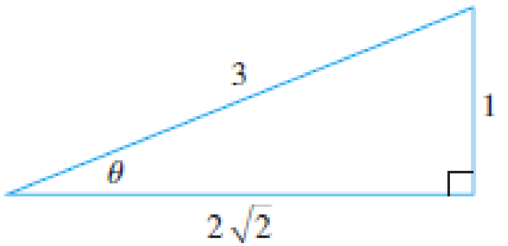
$y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$.

$y = \sec^{-1} x$ is the number in $[0, \pi/2) \cup (\pi/2, \pi]$ for which $\sec y = x$.

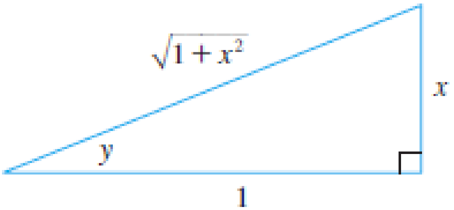
$y = \csc^{-1} x$ is the number in $[-\pi/2, 0) \cup (0, \pi/2]$ for which $\csc y = x$.



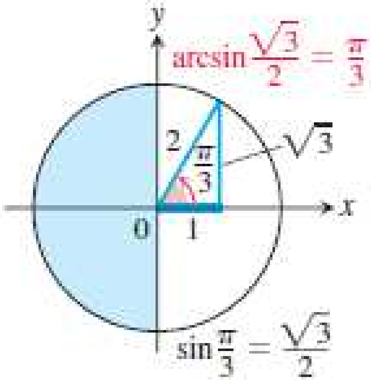


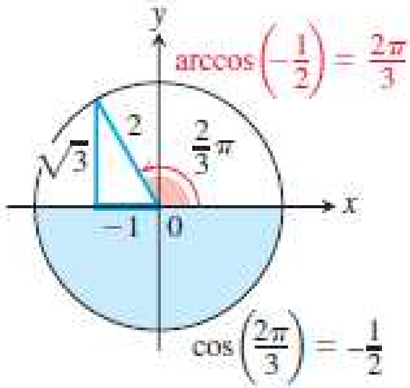


$$\tan\left(\arcsin \tfrac{1}{3}\right) = \tan\theta = \frac{1}{2\sqrt{2}}$$



$$\cos(\tan^{-1} x) = \cos y = \frac{1}{\sqrt{1 + x^2}}$$





Inverse Trigonometric Functions

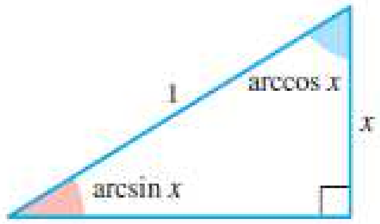


FIGURE 1.69 arcsin x and arccos x are complementary angles (so their sum is $\pi/2$).

$$\arcsin x + \arccos x = \pi/2.$$

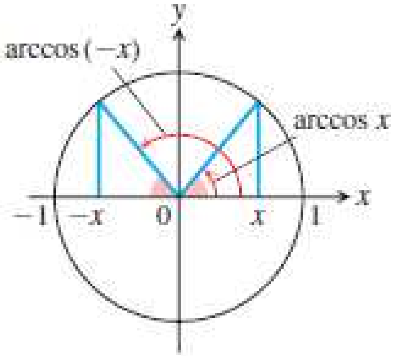


FIGURE 1.68 arccos x and arccos $(-x)$ are supplementary angles (so their sum is π).

$$\arccos x + \arccos(-x) = \pi.$$

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right)$$

$$= \pi/2 - \csc^{-1} x$$

Inverse Trigonometric Functions

The Derivative of $y = \sin^{-1} u$

with $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$:

$$
\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\
&= \frac{1}{\cos(\sin^{-1} x)} && f'(u) = \cos u \\
&= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} && \cos u = \sqrt{1 - \sin^2 u} \\
&= \frac{1}{\sqrt{1 - x^2}}. && \sin(\sin^{-1} x) = x
\end{aligned}
$$

The Derivative of $y = \tan^{-1} u$

$f(x) = \tan x$ and $f^{-1}(x) = \tan^{-1} x$

$$
\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} && \text{Theorem 3} \\
&= \frac{1}{\sec^2(\tan^{-1} x)} && f'(u) = \sec^2 u \\
&= \frac{1}{1 + \tan^2(\tan^{-1} x)} && \sec^2 u = 1 + \tan^2 u \\
&= \frac{1}{1 + x^2}. && \tan(\tan^{-1} x) = x
\end{aligned}
$$

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \qquad |u| < 1.$$

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1 + u^2} \frac{du}{dx}.$$

Inverse Trigonometric Functions

The Derivative of $y = \sec^{-1} u$

$$y = \sec^{-1} x$$

$$\sec y = x \qquad \text{Inverse function relationship}$$

$$\frac{d}{dx}(\sec y) = \frac{d}{dx}x \qquad \text{Differentiate both sides.}$$

$$\sec y \tan y \frac{dy}{dx} = 1 \qquad \text{Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}. \qquad \text{Since } |x| > 1, y \text{ lies in } (0, \pi/2) \cup (\pi/2, \pi) \text{ and } \sec y \tan y \neq 0.$$

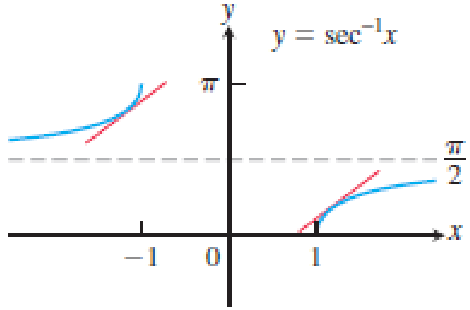


FIGURE 3.44 The slope of the curve $y = \sec^{-1}x$ is positive for both $x < -1$ and $x > 1$.

$$\sec y = x \quad \text{and} \quad \tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$$

$$\frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2 - 1}}.$$

$$\frac{d}{dx}\sec^{-1}x = \begin{cases} +\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x > 1 \\ -\dfrac{1}{x\sqrt{x^2 - 1}} & \text{if } x < -1. \end{cases}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2 - 1}}.$$

$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad |u| > 1.$$

Inverse Trigonometric Functions

Derivatives of the Other Three Inverse Trigonometric Functions

Inverse Function–Inverse Cofunction Identities

$$\cos^{-1} x = \pi/2 - \sin^{-1} x$$
$$\cot^{-1} x = \pi/2 - \tan^{-1} x$$
$$\csc^{-1} x = \pi/2 - \sec^{-1} x$$

TABLE 3.1 Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}, \quad |u| < 1$$
2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}, \quad |u| < 1$$
3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{1}{1 + u^2}\frac{du}{dx}$$
4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{1}{1 + u^2}\frac{du}{dx}$$
5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \quad |u| > 1$$
6. $$\frac{d(\csc^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \quad |u| > 1$$

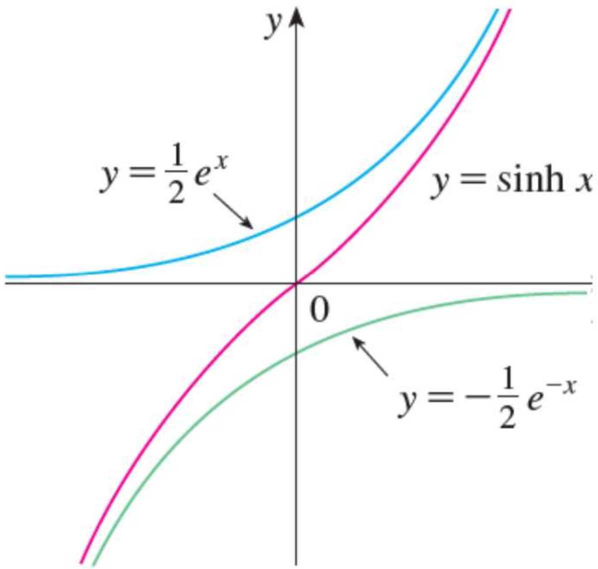
Hyperbolic Functions

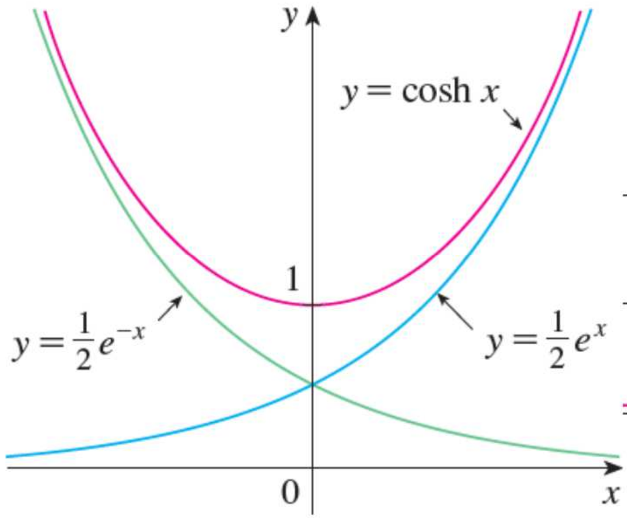
Definition of the Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}$$

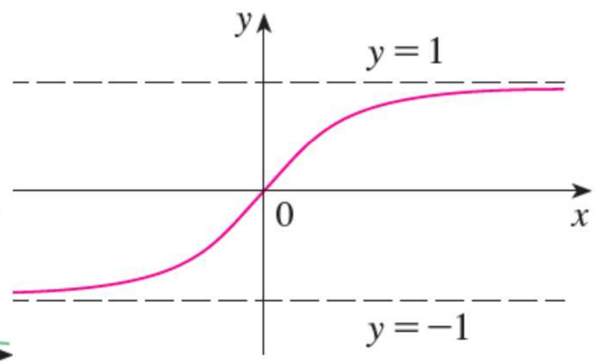
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$





$y = \tanh x$



Hyperbolic Functions

Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1 \qquad 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$= \frac{4}{4} = 1$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x$$

1 Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$$

$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Any of these differentiation rules can be combined with the Chain Rule.
For instance,

$$\frac{d}{dx}\left(\cosh \sqrt{x}\right) = \sinh \sqrt{x} \cdot \frac{d}{dx}\sqrt{x} = \frac{\sinh \sqrt{x}}{2\sqrt{x}}$$

Inverse Hyperbolic Functions

$$y = \sinh^{-1}x \iff \sinh y = x$$

$$y = \cosh^{-1}x \iff \cosh y = x \quad \text{and} \quad y \geq 0$$

$$y = \tanh^{-1}x \iff \tanh y = x$$

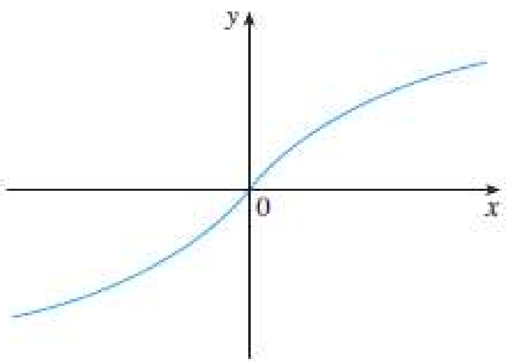


FIGURE 8 $y = \sinh^{-1}x$
domain $= \mathbb{R}$ range $= \mathbb{R}$

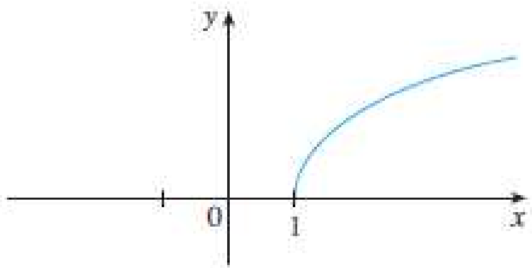


FIGURE 9 $y = \cosh^{-1}x$
domain $= [1, \infty)$ range $= [0, \infty)$

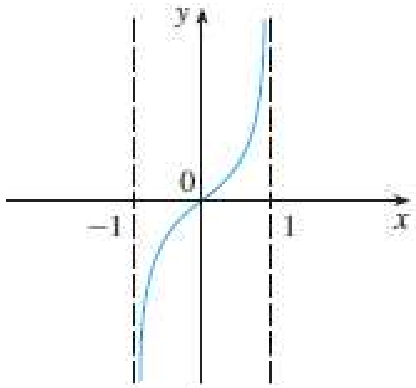


FIGURE 10 $y = \tanh^{-1}x$
domain $= (-1, 1)$ range $= \mathbb{R}$

Inverse Hyperbolic Functions

3	$\sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right)$	$x \in \mathbb{R}$
4	$\cosh^{-1}x = \ln\left(x + \sqrt{x^2-1}\right)$	$x \geqslant 1$
5	$\tanh^{-1}x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$	$-1 < x < 1$

Proof #3: Let $y = \sinh^{-1}x$. Then

$$x = \sinh y = \frac{e^y - e^{-y}}{2}$$

$$e^y - 2x - e^{-y} = 0$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$e^y = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

Note that $e^y > 0$, but $x - \sqrt{x^2+1} < 0$ (because $x < \sqrt{x^2+1}$). Thus the minus sign is inadmissible and we have

$$e^y = x + \sqrt{x^2+1}$$

Therefore $$y = \ln(e^y) = \ln\left(x + \sqrt{x^2+1}\right)$$

This shows that $$\sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right)$$

Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1}x) = \frac{1}{\sqrt{1+x^2}} \quad , \quad \frac{d}{dx}(\cosh^{-1}x) = \frac{1}{\sqrt{x^2-1}} \quad \text{and} \quad \frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$$

EXAMPLE 4 Prove that $\dfrac{d}{dx}(\sinh^{-1}x) = \dfrac{1}{\sqrt{1+x^2}}$.

SOLUTION 1 Let $y = \sinh^{-1}x$. Then $\sinh y = x$.

$$\cosh y \frac{dy}{dx} = 1$$

Since $\cosh^2 y - \sinh^2 y = 1$ and $\cosh y \geqslant 0$,

we have $\cosh y = \sqrt{1+\sinh^2 y}$, so

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$$

SOLUTION 2

$$\begin{aligned}
\frac{d}{dx}(\sinh^{-1}x) &= \frac{d}{dx}\ln\left(x+\sqrt{x^2+1}\right) \\
&= \frac{1}{x+\sqrt{x^2+1}}\frac{d}{dx}\left(x+\sqrt{x^2+1}\right) \\
&= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right) \\
&= \frac{\sqrt{x^2+1}+x}{\left(x+\sqrt{x^2+1}\right)\sqrt{x^2+1}} \\
&= \frac{1}{\sqrt{x^2+1}}
\end{aligned}$$

Thank you for listening.

Moataz El-Zekey