Lecture 2

Compound Stresses

1- Generalized Hooke's law

Generalized Hooke's law is an extension of the simple stress–strain relations of Equation (1.5) to a general case where stresses and strains are three-dimensional.

Consider a cube subjected to normal stresses, σ_x , σ_y and σz , in the directions of x, y, and z coordinate axes, respectively (Figure 1.11(a)).

From Figure 1.11, we have

Strain of Case (a) = strain of Case (b) + strain of Case (c) + strain of Case (d)

In particular, considering the normal strain of Case (a) in the x direction and applying Equation (1.5) and Equation (1.7) to Cases (b), (c) and (d), we have

Normal strain in the x direction		
By σ_x Figure 1.11(b) $\varepsilon_x = \frac{\sigma_x}{E}$	By σ_y Figure 1.11(c) $\nu = -\frac{\varepsilon_x}{\varepsilon_y}$ $\varepsilon_x = -\nu\varepsilon_y$	By σ_z Figure 1.11(d) $\nu = -\frac{\varepsilon_x}{\varepsilon_z}$ $\varepsilon_x = -\nu\varepsilon_z$

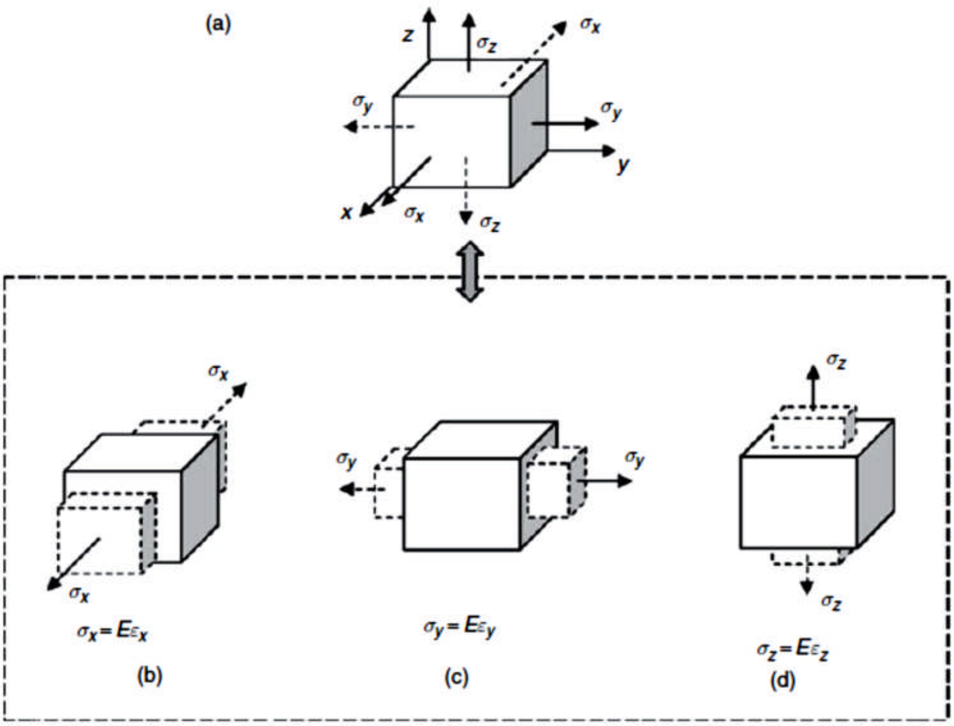

Figure 1.11

Thus, the normal strain of Case (a) in the x direction is as follows:

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\varepsilon_y - \nu\varepsilon_z$$

From Figures 1.11(c) and 1.11(d):

$$\varepsilon_y = \frac{\sigma_y}{E},$$
$$\varepsilon_z = \frac{\sigma_z}{E}$$

Then:

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E} = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \tag{1.8a}$$

The strains in the y and z directions can also be calculated by following exactly the same procedure described above:

$$\begin{aligned}\varepsilon_y &= \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_z}{E} = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]\\ \varepsilon_z &= \frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]\end{aligned} \tag{1.8b}$$

For a three-dimensional case, shear stresses and shear strains may occur within three independent planes, that is, in the x–y, x–z and y–z planes, for which the following three shear stress and strain relations exist:

$$\begin{aligned}\gamma_{xy} &= \frac{\tau_{xy}}{G}\\ \gamma_{xz} &= \frac{\tau_{xz}}{G}\\ \gamma_{yz} &= \frac{\tau_{yz}}{G}\end{aligned} \tag{1.8c}$$

Equation (1.8) is the generalized Hooke's law. The application of Equation (1.8) is limited to isotropic materials in the linear elastic range.

The generalized Hooke's law of Equation (1.8) represents strains in terms of stresses. The following equivalent form of Hooke's law represents stresses in terms of strains:

$$\begin{aligned}\sigma_x &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_x + \frac{\nu}{1-\nu}(\varepsilon_y + \varepsilon_z)\right]\\ \sigma_y &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_y + \frac{\nu}{1-\nu}(\varepsilon_x + \varepsilon_z)\right]\\ \sigma_z &= \frac{E(1-\nu)}{(1+\nu)(1-2\nu)}\left[\varepsilon_z + \frac{\nu}{1-\nu}(\varepsilon_y + \varepsilon_x)\right]\end{aligned} \tag{1.9}$$

$$\begin{aligned}\tau_{xy} &= G\gamma_{xy}\\ \tau_{xz} &= G\gamma_{xz}\\ \tau_{yz} &= G\gamma_{yz}\end{aligned}$$

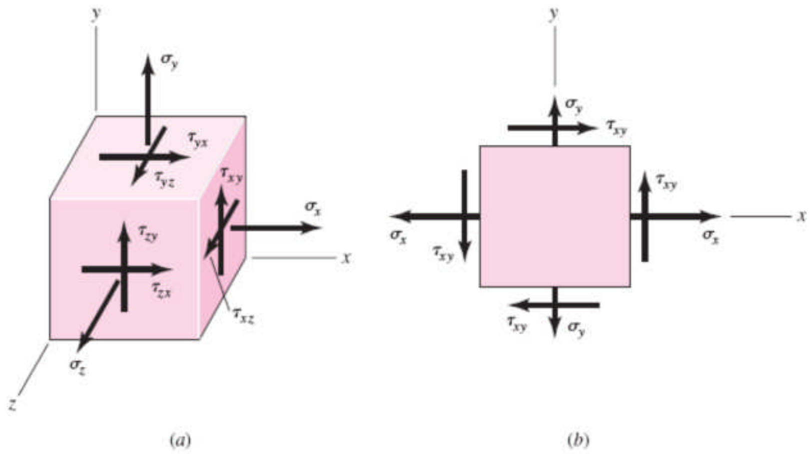

Fig. 1. 12 (*a*) General three-dimensional stress. (*b*) Plane stress with "cross-shears" equal.

2- Mohr's Circle for Plane Stress

Suppose the *dx dy dz* element of Fig.1.12*b* is cut by an oblique plane with a normal *n* at an arbitrary angle φ counterclockwise from the *x* axis as shown in Fig. 1. 13. This section is concerned with the stresses σ and τ that act upon this oblique plane. By summing the forces caused by all the stress components to zero, the stresses σ and τ are found to be

$$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\phi + \tau_{xy}\sin 2\phi \qquad \text{(1-9)}$$

$$\tau = -\frac{\sigma_x - \sigma_y}{2}\sin 2\phi + \tau_{xy}\cos 2\phi \qquad \text{(1-10)}$$

Equations (9) and (10) are called the plane-stress transformation equations. Differentiating Eq. (9) with respect to φ and setting the result equal to zero gives

$$\tan 2\phi_P = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad \text{(1-11)}$$

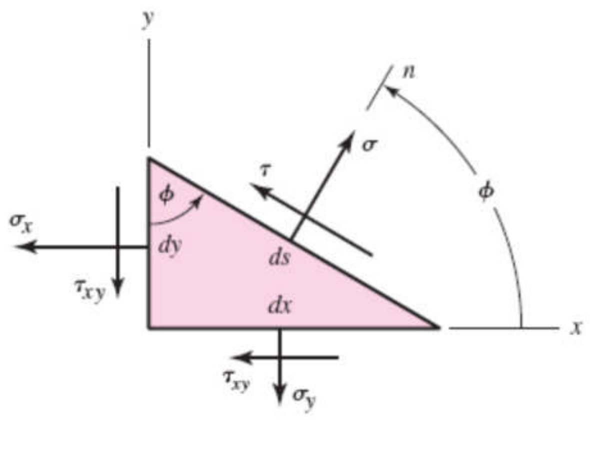

Fig. 1. 13

Equation (11) defines two particular values for the angle $2\varphi_p$, one of which defines the maximum normal stress σ_1 and the other, the minimum normal stress σ_2. These two stresses are called the *principal stresses*, and their corresponding directions, the *principal directions*. The angle between the principal directions is 90°. It is important to note that Eq. (11) can be written in the form

$$\frac{\sigma_x - \sigma_y}{2}\sin 2\phi_P - \tau_{xy}\cos 2\phi_P = 0 \tag{a}$$

Comparing this with Eq. (10), we see that $\tau = 0$, meaning that the *surfaces containing principal stresses have zero shear stresses*.

In a similar manner, we differentiate Eq. (10), set the result equal to zero, and obtain

$$\tan 2\phi_S = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \tag{1-13}$$

Equation (13) defines the two values of $2\phi_s$ at which the shear stress τ reaches an extreme value. The angle between the surfaces containing the maximum shear stresses is 90°. Equation (13) can also be written as

$$\frac{\sigma_x - \sigma_y}{2}\cos 2\phi_P + \tau_{xy}\sin 2\phi_P = 0 \tag{b}$$

Substituting this into Eq. (9) yields

$$\sigma = \frac{\sigma_x + \sigma_y}{2} \tag{1-14}$$

Equation (14) tells us that the two surfaces containing the maximum shear stresses also contain equal normal stresses of $(\sigma_x + \sigma_y)/2$.

Comparing Eqs. (12) and (13), we see that tan *$2\phi_s$* is the negative reciprocal of *tan $2\phi_P$*. This means that *$2\phi_s$* and *$2\phi_p$* are angles 90° apart, and thus the angles between the surfaces containing the maximum shear stresses and the surfaces containing the principal stresses are $\pm 45°$.

Formulas for the two principal stresses can be obtained by substituting the angle *$2\phi_P$* from Eq. (12) in Eq. (10). The result is :

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{1-15}$$

In a similar manner the two extreme-value shear stresses are found to be :

$$\tau_1, \tau_2 = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{1-16}$$

It is important to note that the equations given to this point are quite sufficient for performing any plane stress transformation. However, extreme care must be exercised when applying them.

For example, say you are attempting to determine the principal state of stress for a problem where $\sigma_x = 14$ MPa, $\sigma_y = -10$ MPa, and $\tau_{xy} = -16$ MPa.

Equation (12) yields $\varphi_p = -26.57^\circ$ and 63.43° to locate the principal stress surfaces, whereas, Eq. (15) gives $\sigma_1 = 22$ MPa and $\sigma_2 = -18$ MPa for the principal stresses.

If all we wanted was the principal stresses, we would be finished. However, what if we wanted to draw the element containing the principal stresses properly oriented relative to the *x, y* axes? Well, we have two values of φ_p and two values for the principal stresses. How do we know which value of φ_p corresponds to which value of the principal stress? To clear this up we would need to substitute one of the values of φ_p into Eq. (10) to determine the normal stress corresponding to that angle.

A graphical method for expressing the relations developed in this section, called *Mohr's circle diagram,* is a very effective means of visualizing the stress state at a point and keeping track of the directions of the various components associated with plane stress. Equations (10) and (11) can be shown to be a set of parametric equations for σ and τ , where the parameter is 2φ. The relationship between σ and τ is that of a circle plotted in the σ, τ plane, where the center of the circle is located at

$$C = (\sigma, \tau) = [(\sigma_x + \sigma_y)/2, 0]$$

and has a radius of

$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

A problem arises in the sign of the shear stress. The transformation equations are based on a positive φ being counterclockwise, as shown in Fig. 1. 13. If a positive τ were plotted above the σ axis, points would rotate clockwise on the circle 2φ in the opposite direction of rotation on the element. It would be convenient if the rotations were in the same direction. One could solve the problem easily by plotting positive τ below the axis. However, the classical approach to Mohr's circle uses a different convention for the shear stress.

<u>Mohr's Circle Shear Convention</u>

This convention is followed in drawing Mohr's circle:

- Shear stresses tending to rotate the element clockwise (cw) are plotted *above* the σ axis.
- Shear stresses tending to rotate the element counterclockwise (ccw) are plotted *below* the σ axis.

For example, consider the right face of the element in Fig. 2. 12*b*. By Mohr's circle convention the shear stress shown is plotted *below* the σ axis because it tends to rotate the element counterclockwise. The shear stress on the top face of the element is plotted *above* the σ axis because it tends to rotate the element clockwise.

In Fig. 1.14 we create a coordinate system with normal stresses plotted along the abscissa and shear stresses plotted as the ordinates. On the abscissa, tensile (positive) normal stresses are plotted to the right of the origin *O* and compressive (negative) normal stresses to the left. On the ordinate, clockwise (cw) shear stresses are plotted up; counterclockwise (ccw) shear stresses are plotted down.

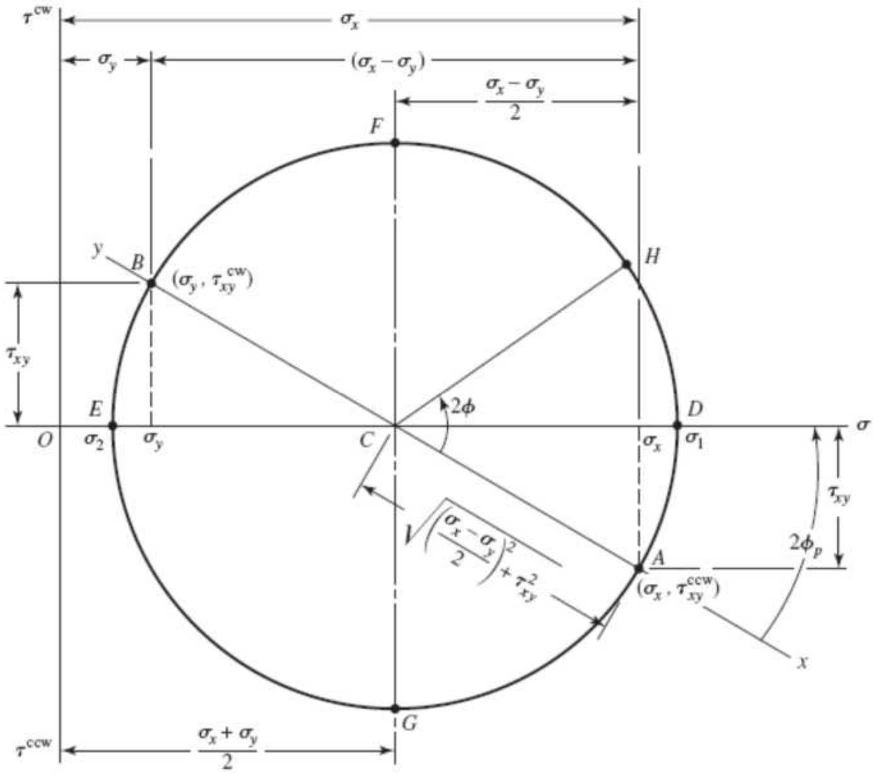

Fig. 1. 14 Mohr's circle diagram.

Using the stress state of Fig. 1. 12*b*, we plot Mohr's circle, Fig. 1. 14, by first looking at the right surface of the element containing σ_x to establish the sign of σ_x and the cw or ccw direction of the shear stress. The right face is called the *x face* where φ = 0◦. If σ_x is positive and the shear stress τxy is ccw as shown in Fig. 1. 12*b*, we can establish point *A* with coordinates (σ_x , τ_{xy}^{ccw}) in Fig. 1. 14. Next, we look at the top *y face,* where φ = 90◦, which contains σ_y , and repeat the process to obtain point *B* with coordinates (σy , τ_{xy}^{cw}) as shown in Fig. 2. 9. The two states of stress for the element are Δφ = 90◦ from each other on the element so they will be 2Δφ = 180◦ from each other on Mohr's circle. Points *A* and *B* are the same vertical distance from the σ axis.

Thus, *AB* must be on the diameter of the circle, and the center of the circle *C* is where *AB* intersects the σ axis. With points *A* and *B* on the circle, and center *C,* the complete circle can then be drawn. Note that the extended ends of line *AB* are labeled *x* and *y* as references to the normals to the surfaces for which points *A* and *B* represent the stresses.

The entire Mohr's circle represents the state of stress at a *single* point in a structure. Each point on the circle represents the stress state for a *specific* surface intersecting the point in the structure. Each pair of points on the circle 180° apart represent the state of stress on an element whose surfaces are 90° apart. Once the circle is drawn, the states of stress can be visualized for various surfaces intersecting the point being analyzed.

For example, the principal stresses σ_1 and σ_2 are points *D* and *E*, respectively, and their values obviously agree with Eq. (15). We also see that the shear stresses are zero on the surfaces containing σ_1 and σ_2. The two extreme-value shear stresses, one clockwise and one counterclockwise, occur at *F* and *G* with magnitudes equal to the radius of the circle. The surfaces at *F* and *G* each also contain normal stresses of $(\sigma_x + \sigma_y)/2$ as noted earlier in Eq. (15). Finally, the state of stress on an arbitrary surface located at an angle φ counterclockwise from the *x* face is point *H*.

At one time, Mohr's circle was used graphically where it was drawn to scale very accurately and values were measured by using a scale and protractor. Here, we are strictly using Mohr's circle as a visualization aid and will use a semi graphical approach, calculating values from the properties of the circle. This is illustrated by the following example.

EXAMPLE 1

A stress element has σ_x = 80 MPa and τ_{xy} = 50 MPa cw, as shown in Fig. 1. 15*a*.

(*a*) Using Mohr's circle, find the principal stresses and directions, and show these on a stress element correctly aligned with respect to the *xy* coordinates. Draw another stress element to show τ_1 and τ_2, find the corresponding normal stresses, and label the drawing completely.

Solution

In the semi graphical approach used here, we first make an approximate freehand sketch of Mohr's circle and then use the geometry of the figure to obtain the desired information.

Draw the σ and τ axes first (Fig. 1. 15*b*) and from the *x* face locate σ_x = 80 MPa along the σ axis. On the *x* face of the element, we see that the shear stress is 50 MPa in the cw direction. Thus, for the *x* face, this establishes point *A* (80, 50cw) MPa. Corresponding to the *y* face, the stress is σ = 0 and τ = 50 MPa in the ccw direction.

This locates point *B* (0, 50ccw) MPa. The line *AB* forms the diameter of the required circle, which can now be drawn. The intersection of the circle with the σ axis defines σ_1 and σ_2 as shown. Now, noting the triangle *ACD*, indicate on the sketch the length of the legs *AD* and *CD* as 50 and 40 MPa, respectively. The length of the hypotenuse *AC* is

$$\tau_1 = \sqrt{(50)^2 + (40)^2} = 64.0 MPa$$

and this should be labeled on the sketch too. Since intersection C is 40 *MPa* from the origin, the principal stresses are now found to be

$\sigma_1 = 40 + 64 = 104\ MPa$ and $\sigma_2 = 40 - 64 = -24\ MPa$

The angle 2φ from the *x* axis *cw* to σ_1 is

$$2\phi_P = \tan^{-1}\frac{50}{40} = 51.3^\circ$$

To draw the principal stress element (Fig. 1. 15*c*), sketch the *x* and *y* axes parallel to the original axes. The angle φ_p on the stress element must be measured in the *same* direction as is the angle $2\varphi_p$ on the Mohr circle. Thus, from *x* measure 25.7° (half of 51.3°) clockwise to locate the σ_1 axis. The σ2 axis is 90° from the σ_1 axis and the stress element can now be completed and labeled as shown. Note that there are *no* shear stresses on this element.

The two maximum shear stresses occur at points *E* and *F* in Fig. 1. 15*b*. The two normal stresses corresponding to these shear stresses are each 40 MPa, as indicated.

Point *E* is 38.7° ccw from point *A* on Mohr's circle. Therefore, in Fig. 1. 15*d*, draw a stress element oriented 19.3° (half of 38.7°) ccw from *x*. The element should then be labeled with magnitudes and directions as shown.

In constructing these stress elements it is important to indicate the *x* and *y* directions of the original reference system. This completes the link between the original machine element and the orientation of its principal stresses.

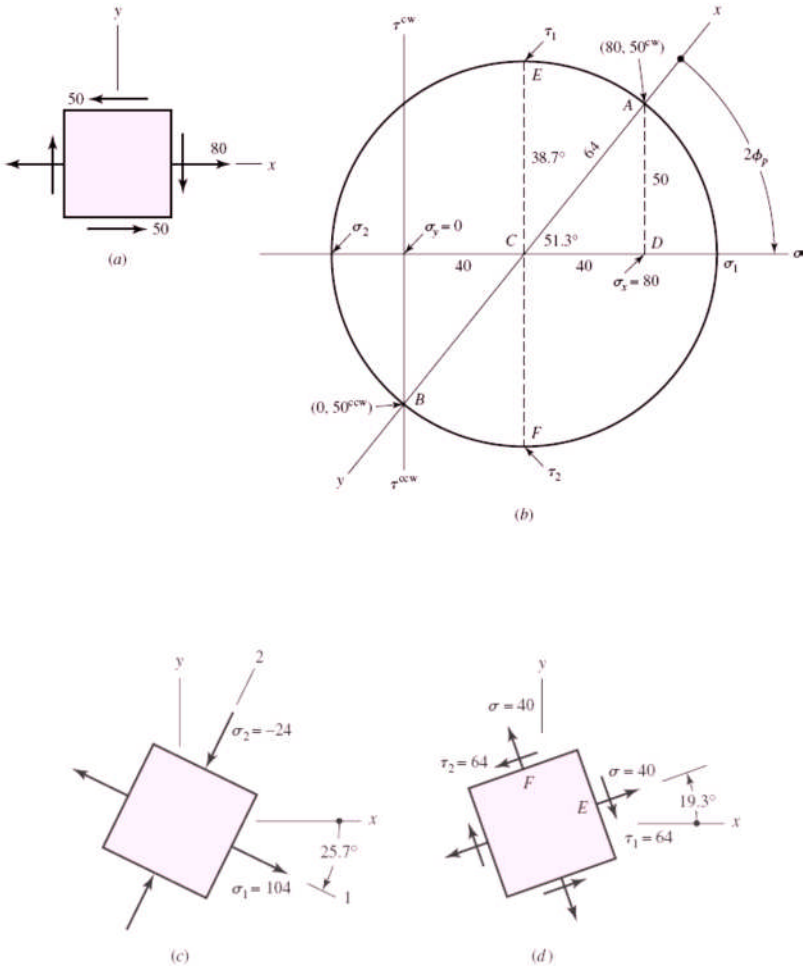

Fig. 1. 15 All stresses in *MPa*.

Table 1.1. Conversion Factors

Quantity	Symbol	SI Units	English Units	To Convert from English to SI Units Multiply by
Length	L	m	ft	0.3048
Mass	m	kg	lbm	0.4536
Time	t	s	sec	1
Area	A	m^2	ft^2	0.09290
Volume	V	m^3	ft^3	0.02832
Velocity	V	m/s	ft/sec	0.3048
Acceleration	a	m/s^2	ft/sec^2	0.3048
Angular velocity	ω	rad/s	rad/sec	1
		rad/s	rpm	9.55
Force, Weight	F, W	N	lbf	4.448
Density	ρ	kg/m^3	lbm/ft^3	16.02
Specific weight	γ	N/m^3	lbf/ft^3	157.1
Pressure, stress	ρ, σ, τ	kPa	psi	6.895
Work, Energy	W, E, U	J	ft-lbf	1.356
Power	W	W	ft-lbf/sec	1.356
		W	hp	746

Table 1.2. Prefixes for SI Units

Multiplication Factor	Prefix	Symbol
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{-2}	centi*	c
10^{-3}	mili	m
10^{-6}	micro	µ
10^{-9}	nano	n
10^{-12}	pico	p

*Discouraged except in cm, cm^2, cm^3, or cm^4.