LECTURE NOTES

KINEMATICS OF FLUID MOTION 2

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Continuity equation for one-dimensional flow

- Therefore the continuity equation of steady flow :

$$\rho_1 A_1 v_1 = \rho_2 A_2 v_2$$

Interpretation : The mass flow rate $\dot{m} = \rho A v = const.$

through a steady stream-tube or a duct.

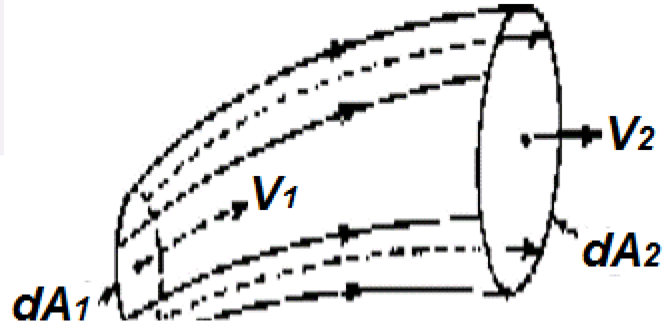
- For incompressible fluid with $\rho_1 = \rho_2$:

$$A_1 v_1 = A_2 v_2$$

Interpretation : The volume flow rate $\dot{V} = A v = const.$

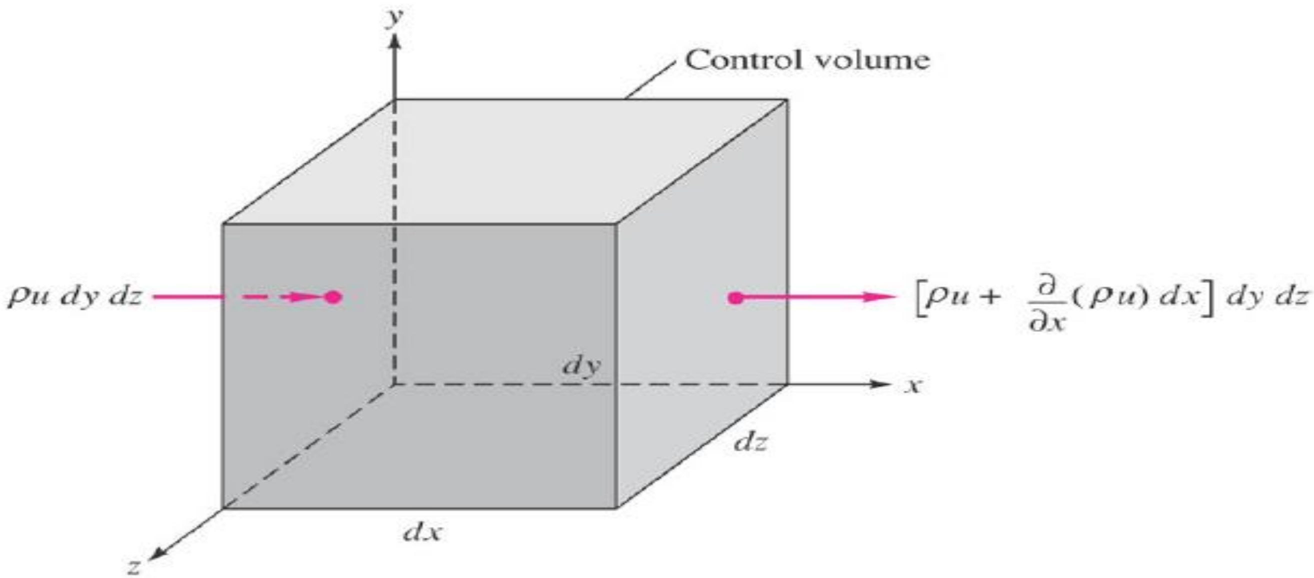
- From the continuity equation for incompressible fluid :

$$\frac{v_1}{v_2} = \frac{A_2}{A_1}$$ for a stream-tube.



Flow contiuity, through a stream tube

Continuity equation in Cartesian coordinates
(The differential equation of mass conservation)



Fixed Cartesian element showing inlet and outlet flows on the x-direction.

Face	Inlet mass flow	Outlet mass flow
x	$\rho u\, dy\, dz$	$\left[\rho u + \frac{\partial}{\partial x}(\rho u)\, dx\right] dy\, dz$
y	$\rho v\, dx\, dz$	$\left[\rho v + \frac{\partial}{\partial y}(\rho v)\, dy\right] dx\, dz$
z	$\rho w\, dx\, dy$	$\left[\rho w + \frac{\partial}{\partial z}(\rho w)\, dz\right] dx\, dy$

$$\int_{CV} \frac{\partial \rho}{\partial t}\, d\mathcal{V} + \sum_i (\rho_i A_i V_i)_{out} - \sum_i (\rho_i A_i V_i)_{in} = 0$$

$$\int_{CV} \frac{\partial \rho}{\partial t}\, d\mathcal{V} \approx \frac{\partial \rho}{\partial t}\, dx\, dy\, dz$$

$$\frac{\partial \rho}{\partial t}\, dx\, dy\, dz + \frac{\partial}{\partial x}(\rho u)\, dx\, dy\, dz + \frac{\partial}{\partial y}(\rho v)\, dx\, dy\, dz + \frac{\partial}{\partial z}(\rho w)\, dx\, dy\, dz = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z}$$

$$= \frac{\partial \rho}{\partial t} + \nabla.(\rho V) = 0$$

For steady compressible flow, continuity equation simplifies to:

$$\text{Cartesian,} \qquad \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

For incompressible flow, continuity equation can be further simplified since density changes are negligible:

$$\nabla . V = 0$$

$$\text{Cartesian,} \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Example

A pipeline carries oil (sp. gr. 0.83), at a velocity of 2 m/s through a 20 cm pipe. At another section the diameter is 15 cm.. Find the velocity at this section and the mass rate of flow.

Sol. The continuity equation applied to the two sections under consideration yields :

$$Q = A_1.V_1 = A_2.V_2$$

$$V_2 = A_1.V_1 / A_2 = ((\pi/4) \times (0.2)^2 \times 2) / ((\pi/4) \times (0.15)^2) = 3.56 \text{ m/s.}$$

Volume rate of flow = $Q = A_1 V_1 = (\pi/4) \times (0.2)^2 \times 2 = 0.0629 \text{ m}^3\text{/sec}$

Mass rate of flow $\rho Q = 1000 \times 0.83 \times 0.0629 = 52.2$ kg/s.

Example

A pipeline 60 cm in diameter bifurcates at a Y-junction into two branches 40 cm and 30 cm in diameter. If the rate of flow in the main pipe is 1.5 m^3/s, and the mean velocity of flow in the 30 cm pipe is 7.5 m/s, determine the rate of flow in the 40 cm pipe.

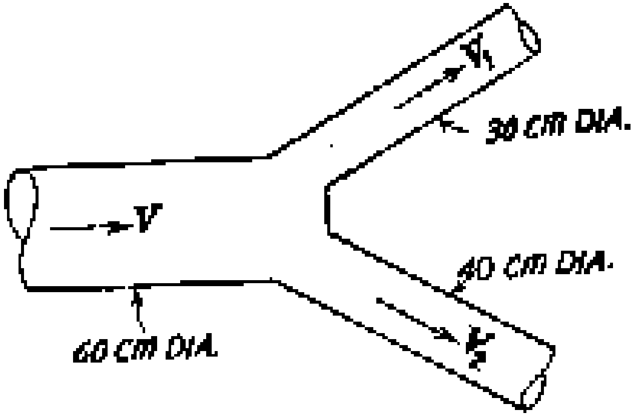
Sol. From the continuity relationship,

$Q = Q_1 + Q_2 = A_1V_1 + A_2V_2.$

Rate of flow in 30 cm pipe

$= (\pi/4) \times (0.3)^2 x7.5 = 0.53\ m^3/s$

$Q_2 = Q - Q_1 = 1.5 – 0.53 = 0.97\ m^3/s$



Example

An idealized incompressible flow has the proposed three-dimensional velocity distribution

$$\mathbf{V} = 4xy^2\mathbf{i} + f(y)\mathbf{j} - zy^2\mathbf{k}$$

Solution: **Simply substitute the given velocity components into the incompressible continuity equation:**

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{\partial}{\partial x}(4xy^2) + \frac{\partial f}{\partial y} + \frac{\partial}{\partial z}(-zy^2) = 4y^2 + \frac{df}{dy} - y^2 = 0$$

$$or: \quad \frac{df}{dy} = -3y^2. \quad \text{Integrate: } f(y) = \int(-3y^2)dy = -\mathbf{y}^3 + \mathbf{constant}$$

Example

Which of the following velocity fields satisfies conservation of mass for incompressible plane flow?

(a) $u = -x,\ v = y$ **(b)** $u = 3y,\ v = 3x$ **(c)** $u = 4x,\ v = -4y$

(d) $u = 3xt,\ v = 3yt$ **(e)** $u = xy + y^2 t,\ v = xy + x^4 t$ **(f)** $u = 4x^2 y^3,\ v = -2xy^4$

Ignore dimensional inconsistencies.

In order to satisfy continuity,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial t} = 0 \qquad \text{or} \qquad \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y}$$

(a) $\frac{\partial u}{\partial x} = -1$ and $\frac{\partial v}{\partial y} = 1$ therefore, it does satisfy continuity.

(b) $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$ therefore, it does satisfy continuity.

(c) $\frac{\partial u}{\partial x} = 4$ and $\frac{\partial v}{\partial y} = -4$ therefore, it does satisfy continutiy.

(d) $\frac{\partial u}{\partial x} = 3t$ and $\frac{\partial v}{\partial y} = 3t$ therefore, it does not satisfy continuity.

(e) $\frac{\partial u}{\partial x} = y$ and $\frac{\partial v}{\partial y} = x$ therefore, it does not satisfy continuity.

(f) $\frac{\partial u}{\partial x} = 8xy^3$ and $\frac{\partial v}{\partial y} = -8xy^3$ therefore, it does satisfy continuity.

Example

After discarding any constants of integration, determine the appropriate value of the unknown velocities *u or v which satisfy the equation of two-dimensional incompressible* continuity for:

$$\text{(a) } u = x^2 y; \quad \text{(b) } v = x^2 y; \quad \text{(c) } u = x^2 - xy; \quad \text{(d) } v = y^2 - xy$$

Solution: Substitute the given component into continuity and solve for the unknown component:

$$\text{(a)} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 = \frac{\partial}{\partial x}(x^2 y) + \frac{\partial v}{\partial y}; \quad \frac{\partial v}{\partial y} = -2xy, \quad \textbf{or:} \quad \mathbf{v = -xy^2 + f(x)} \quad \textit{Ans.} \text{ (a)}$$

$$\text{(b)} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 = \frac{\partial u}{\partial x} + \frac{\partial}{\partial y}(x^2 y); \quad \frac{\partial u}{\partial x} = -x^2, \quad \textbf{or:} \quad \mathbf{u = -\frac{x^3}{3} + f(y)} \quad \textit{Ans.} \text{ (b)}$$

$$\text{(c)} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 = \frac{\partial}{\partial x}(x^2 - xy) + \frac{\partial v}{\partial y}; \quad \frac{\partial v}{\partial y} = -2x + y, \quad \textbf{or:} \quad \mathbf{v = -2xy + \frac{y^2}{2} + f(x)}$$

$$\text{(d)} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 = \frac{\partial u}{\partial x} + \frac{\partial}{\partial y}(y^2 - xy); \quad \frac{\partial u}{\partial x} = -2y + x \quad \textbf{or:} \quad \mathbf{u = -2xy + \frac{x^2}{2} + f(y)}$$

Irrotational Flow

- An irrotational flow is one in which fluid elements moving in the flow field do not undergo any rotation. For

$$\vec{\omega} = 0, \nabla \times \vec{V} = 0$$

The *rotation, of the element about the z axis is:*

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \qquad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

Rotation of the field element about the other two coordinate axes

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$$

Vorticity

The *vorticity*, ζ, is defined as a vector that is twice the rotation vector; that is,

$$\boldsymbol{\zeta} = 2\,\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{V}$$

Example

GIVEN For a certain two-dimensional flow field the velocity is given by the equation

$$\mathbf{V} = (x^2 - y^2)\hat{\mathbf{i}} - 2xy\hat{\mathbf{j}}$$

FIND Is this flow irrotational?

Solution

For an irrotational flow the rotation vector, $\boldsymbol{\omega}$, having the components given by Eqs. 6.12, 6.13, and 6.14 must be zero. For the prescribed velocity field

$$u = x^2 - y^2 \qquad v = -2xy \qquad w = 0$$

and therefore

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = 0$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = 0$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}\left[(-2y) - (-2y)\right] = 0$$

Thus, the flow is irrotational. (Ans)

STREAM FUNCTION

- The concept of stream function ψ is based on the principle of continuity and the properties of a stream line, It relates the streamlines and mass flow rate in **2-D, incompressible flow.**
- An equation that would describe such streamlines in a 2D flow may be write $\psi = \psi(x, y)$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

the definition of a streamline in two-dimensional flow is

$$\frac{dx}{u} = \frac{dy}{v}$$

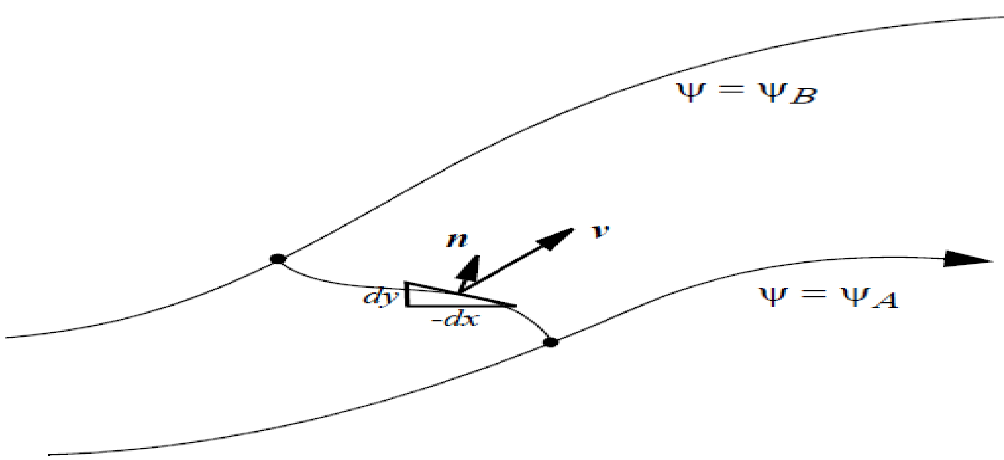
or $\quad u\,dy - v\,dx = 0 \quad$ streamline

Introducing the stream function , we have

$$\frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy = 0 = d\psi$$

Thus the change in ψ is zero along a streamline, or

$$\psi = \text{const along a streamline}$$



$$Q_{AB} = \int_A^B (u\,dy - v\,dx) = \int_A^B d\psi$$

$$Q_{AB} = \psi_B - \psi_A$$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

continuity equation $\partial u/\partial x + \partial \upsilon/\partial y = 0$

substituting .the values of u and υ in terms of ψ

$$\partial^2 \psi / \partial x \partial y - \partial^2 \psi / \partial y \partial x = 0$$

since ψ, is continuous function of x and y. it may be observed that **the stream function ψ satisfies the continuity equation**

For irrotational flow,

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$\omega_z = 0$

and, therefore, $\partial \upsilon / \partial x = \partial u / \partial y$

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

expressing u and υ in terms of ψ ,we get

$$-\partial^2\psi/\partial x^2 = \partial^2\psi/\partial y^2$$

Or $$\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2 = 0$$

This equation is Laplace’s equation for irrotational flow

This differential equation arises in many different areas of engineering and physics and is called *Laplace’s equation. Thus, inviscid, incompressible, irrotational flow fields are governed by* Laplace’s equation.

VELOCITY POTENTIAL

We can formulate a relation called the potential function, φ, for a velocity field that is irrotational. The velocity components can be expressed in terms of a scalar function

φ(*x, y, z, t*) *as:*

$$u = -\partial \varphi / \partial x \quad \text{and} \quad v = -\partial \varphi / \partial y, \quad w = -\partial \varphi / \partial z$$

The continuity equation for three-dimensional steady flow of incompressible fluids , gives

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$$

which may be written in terms of φ as

$$\partial^2\varphi/\partial x^2 + \partial^2\varphi/\partial y^2 + \partial^2\varphi/\partial z^2 = 0$$

This equation is **Laplace's equation for irrotational flow**

For irrotational flow,

$\omega_z = 0$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

and, therefore, $\partial \upsilon/\partial x = \partial u/\partial y$

expressing u and υ in terms of φ ,we get

$$\partial^2\varphi/\partial x\,\partial y = \partial^2\varphi/\partial y\,\partial x$$

Slope of stream lines and Potential lines

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = 0$$

$$u = \frac{\partial \Psi}{\partial y} \qquad v = -\frac{\partial \Psi}{\partial x}$$

The slope of a streamline—a line of constant ψ—is given by

$$\left.\frac{dy}{dx}\right)_{\psi} = -\frac{\partial \psi / \partial x}{\partial \psi / \partial y} = -\frac{-v}{u} = \frac{v}{u}$$

Along a line of constant ϕ, $d\phi = 0$ and

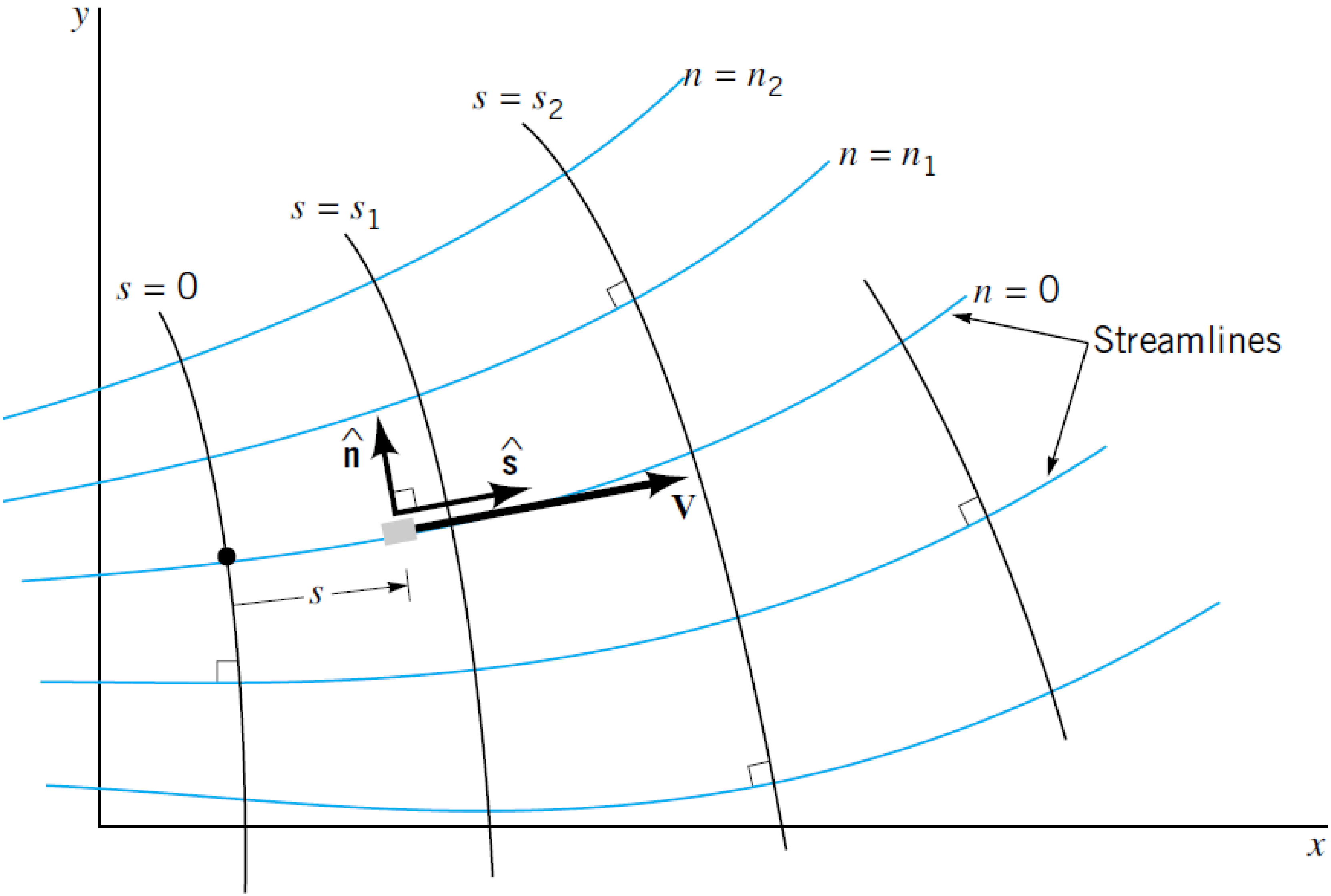
u = - ∂ φ /∂ x and v = - ∂ φ /∂y

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0$$

Consequently, the slope of a potential line—a line of constant ϕ—is given by

$$\left.\frac{dy}{dx}\right)_{\phi} = -\frac{\partial \phi / \partial x}{\partial \phi / \partial y} = -\frac{u}{v}$$

constant ψ line at any point is the negative reciprocal of the slope of the constant φ line at that point; **lines of constant ψ and constant φ are orthogonal.** **This property of potential lines and** streamlines is useful in graphical analyses of flow fields.



y
x
s = 0
s = s₁
s = s₂
n = n₂
n = n₁
n = 0
Streamlines
n̂
ŝ
V
s

Example

If φ = 3xy, find x and y components of velocity at (1,3) and (3,3). Determine the discharge passing between streamlines passing these points.

Sol. velocity components in terms of φ are given by

$u = -\partial\varphi / \partial x$ and $v = -\partial\varphi / \partial y$

$\varphi = 3xy$ $\quad\quad$ $\partial\varphi / \partial x = 3y$ and $\partial\varphi / \partial y = 3x$

hence the velocity components $\quad$ **$u = -3y$** and $\quad$ **$v = -3x$**

The velocity components at (1,3) are $u = -9$ and $v = -3$

and at (3,3) , $u = -9$ and $v = -9$

The total derivative ψ may be written as

$d\psi = \partial\psi/\partial x \; dx + \partial\psi/\partial y \; dy$

but $\quad\quad$ $u = \partial\psi / \partial y$ and $v = -\partial\psi / \partial x$

the total derivative $\quad\quad$ $d\psi = -v\,dx + u\,dy = 3x\,dx + (-3y)\,dy$

Integrating $\psi = 3/2\,x^2 - 3/2 y^2 + A$ $\quad$ where A = constant of integration.

Discharge between the streamlines passing through (1,3) and (3,3)

$$= \psi_{(1,3)} - \psi_{(3,3)} = 3/2\,(1-9) - 3/2\,(9-9) = -12 \text{ units.}$$

Example

For an incompressible flow represented by $\psi = x^2 - y^2$, calculate the total acceleration vector and show that it is proportional to the radius vector.

Sol. The total acceleration $$u = \frac{\partial \Psi}{\partial y} \qquad v = -\frac{\partial \Psi}{\partial x}$$

$a = \sqrt{(a_x^2 + a_y^2)}$

also $$\mathbf{a_x = u\, \partial u/\partial x + v\, \partial u/\partial y}$$
$$\mathbf{a_y = u\, \partial v/\partial x + v\, \partial v/\partial y}$$

For the given stream function , $\psi = x^2 - y^2$

$\mathbf{u} = \partial\psi/\partial y = \mathbf{-2y}$ and $\mathbf{v} = -\partial\psi/\partial x = \mathbf{-2x}$

$\mathbf{\partial u/\partial x = 0,\ \partial u/\partial y = -2,\ \partial v/\partial x = -2,\ \partial v/\partial y = 0}$

$$\mathbf{a_x = 0 + -2x\,(-2) = 4x}$$

and $$\mathbf{a_y = (-2y)(-2) + 0 = 4y}$$

$a = \sqrt{(a_x^2 + a_y^2)} = 4\sqrt{(x^2 + y^2)} = 4r$

Since acceleration $a = 4r$, it is proportional to the radius r.