Lecture 8
Taylor and Laurent series

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Complex sequences

- An ***infinite sequence*** of complex numbers $z_1, z_2, \cdots, z_n, \cdots$, denoted by $\{z_n\}$, can be considered as a function defined on a set of positive integers into the complex plane.

- For example, we take $z_n = \frac{n+i}{2^n}$ so that the complex sequence is

$$\{z_n\} = \left\{\frac{1+i}{2}, \frac{2+i}{2^2}, \frac{3+i}{2^3}, \cdots\right\}$$

- Convergence of complex sequences

  - Given a complex sequence $\{z_n\}$, if for each positive quantity ϵ, there exists a positive integer N such that

$$|z_n - z| < \epsilon \quad \text{whenever } n > N,$$

then the sequence is said to ***converge*** to the limt z. We write

$$\lim_{n \to \infty} z_n = z.$$

If there is no limit, we say that the sequence ***diverges***.

Complex sequences

- It is easy to show that

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \quad \text{and} \quad \lim_{n\to\infty} y_n = y.$$

  - Therefore, the study of the convergence of a complex sequence is equivalent to the consideration of two real sequences.

- The above theorem enables us to write

$$\lim_{n\to\infty}(x_n + iy_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

whenever we know that both limits on the right exist or the one on the left exists.

For example, the sequence

$$z_n = \frac{1}{n^3} + i, \quad n = 1, 2, \cdots,$$

converges to i since $\lim_{n\to\infty} \frac{1}{n^3}$ and $\lim_{n\to\infty} 1$ exist, so

$$\lim_{n\to\infty}\left(\frac{1}{n^3} + i\right) = \lim_{n\to\infty} \frac{1}{n^3} + i \lim_{n\to\infty} 1 = 0 + i \cdot 1 = i.$$

Infinite series of complex numbers

- An ***infinite series*** of complex numbers $z_1, z_2, \cdots, z_n, \cdots$ is the infinite sum of the sequence $\{z_n\}$ given by

$$z_1 + z_2 + z_3 + \cdots = \lim_{n\to\infty} \left(\sum_{k=1}^{n} z_k \right).$$

- To study the properties of an infinite series, we define the sequence of ***partial sums*** $\{S_n\}$ by

$$S_n = \sum_{k=1}^{n} z_k.$$

- If the limit of the sequence $\{S_n\}$ converges to S, then the series is said to be ***convergent*** and S is its ***sum***; otherwise, the series is ***divergent***.

- The sum, when it exists, is unique.

- The consideration of an infinite series is relegated to that of an infinite sequence of partial sums.

Infinite series of complex numbers

❑ Convergence of Complex Series

> **Theorem 6.2** Suppose that $z_n = x_n + iy_n$ ($n = 1, 2, \ldots$) and $S = X + iY$. Then
>
> $$\sum_1^\infty z_n = S \Leftrightarrow \sum_1^\infty x_n = X \text{ and } \sum_1^\infty y_n = Y.$$

- Hence, $\Sigma\, z_n$ is convergent if and only if $\Sigma\, x_n$ and $\Sigma\, y_n$ are convergent.
- There are many parallels with real series.
- Now if $\Sigma\, x_n$ and $\Sigma\, y_n$ are convergent, then $x_n \to 0$, $y_n \to 0$. We deduce that $\Sigma\, z_n$ convergent $\Longrightarrow z_n \to 0$. Of course, the converse is false!!.
- Hence, A necessary condition for the convergence of a complex series is that

$$\lim_{n\to\infty} z_n = 0.$$

- So the terms of a convergent complex sequence are bounded: that is, there exists M: $|\, z_n \,| < M$ for all n.

Infinite series of complex numbers

❑ Absolute convergence

The complex series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |z_n|$ converges. Note that $|z_n| = \sqrt{x_n^2 + y_n^2}$ and since

$$|x_n| \leq \sqrt{x_n^2 + y_n^2} \quad \text{and} \quad |y_n| \leq \sqrt{x_n^2 + y_n^2},$$

then from the comparison test, the two series

$$\sum_{n=1}^{\infty} |x_n| \quad \text{and} \quad \sum_{n=1}^{\infty} |y_n|$$

must converge. Thus, absolute convergence in a complex sequence implies convergence in that sequence.

The converse may not hold. If Σz_n converges but $\Sigma |z_n|$ does not, the series Σz_n is said to be *conditionally convergent*. For example,

$$-\text{Log}(1 - e^{i\theta}) = \sum_{n=1}^{\infty} \frac{e^{in\theta}}{n}, \quad \theta \neq 0,$$

is conditionally convergent.

Infinite series of complex numbers

Example

Show that the series $\sum_{j=1}^{\infty} (3+2i)/(j+1)^j$ converges.

Solution

We compare the series

$$\sum_{j=1}^{\infty} \frac{3+2i}{(j+1)^j} = \frac{(3+2i)}{9} + \frac{(3+2i)}{64} + \cdots \qquad (A)$$

with the *convergent* geometric series

$$\sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots . \qquad (B)$$

Since $|3+2i| = \sqrt{13} < 4$, one can easily verify that for $j \geq 3$

$$\left| \frac{3+2i}{(j+1)^j} \right| < \frac{4}{(j+1)^j} \leq \frac{1}{2^j}.$$

The terms of (B) dominate those of (A), hence (A) converges.

Sequences of complex functions

Let $f_1(z), \cdots, f_n(z), \cdots$, denoted by $\{f_n(z)\}$, be a sequence of complex functions of z that are defined and single-valued in a region R in the complex plane.

For some point $z_0 \in R, \{f_n(z_0)\}$ becomes a sequence of complex numbers. Supposing $\{f_n(z_0)\}$ converges, the limit is unique. The value of the limit depends on z_0, and we write

$$f(z_0) = \lim_{n\to\infty} f_n(z_0).$$

If this holds for every $z \in R$, the sequence $\{f_n(z)\}$ defines a complex function $f(z)$ in R. We write

$$f(z) = \lim_{n\to\infty} f_n(z).$$

This is usually called *pointwise convergence*.

The region R is called the *region of convergence* of the sequence of complex functions.

Convergence of series of complex functions

An infinite series of complex functions

$$f_1(z) + f_2(z) + f_3(z) + \cdots = \sum_{k=1}^{\infty} f_k(z)$$

is related to the sequence of partial sum $\{S_n(z)\}$

$$S_n(z) = \sum_{k=1}^{n} f_k(z).$$

The infinite series is said to be *convergent* if

$$\lim_{n\to\infty} S_n(z) = S(z),$$

where $S(z)$ is called the *sum*; otherwise the series is *divergent*.

Example. Show that the series

$$z(1-z) + z^2(1-z) + z^3(1-z) + \cdots$$

converges for $|z| < 1$ and find its sum.

Solution

$$\begin{aligned} S_n(z) &= z(1-z) + \cdots + z^n(1-z) \\ &= z - z^2 + z^2 - z^3 + \cdots + z^n - z^{n+1} = z - z^{n+1}. \end{aligned}$$

Hence, $\lim_{n\to\infty} S_n(z) = z$ for all z such that $|z| < 1$.

Convergence of series of complex functions

Many of the properties related to convergence of complex functions can be extended from their counterparts of complex numbers. For example, a necessary but not sufficient condition for the infinite series of complex functions to converge is that

$$\lim_{k\to\infty} f_k(z) = 0,$$

for all z in the region of convergence.

- Weierstrass *M*-test

  If $|f_k(z)| \leq M_k$ where M_k is independent of z in $\mathcal{R}$ and the series ΣM_k converges, then $\Sigma f_k(z)$ is convergent in $\mathcal{R}$.

Convergence of series of complex functions

- Test for convergence using the *M*-test

1. $\sum_{n=1}^{\infty} \frac{z^n}{n\sqrt{n+1}}, |z| \leq 1.$

   Note that $|f_n(z)| = \frac{|z|^n}{n\sqrt{n+1}} \leq \frac{1}{n^{3/2}}$ if $|z| \leq 1$. Take $M_n = \frac{1}{n^{3/2}}$ and note that $\sum M_n$ converges.

2. $\sum_{n=1}^{\infty} \frac{1}{n^2+z^2}, 1 < |z| < 2.$

   Omit the first two terms since it does not affect the convergence property. For $n \geq 3$ and $1 < |z| < 2$, we have

$$|n^2+z^2| \geq |n^2| - |z^2| \geq n^2 - 4 \geq \frac{n^2}{2} \text{ so that } \left|\frac{1}{n^2+z^2}\right| \leq \frac{2}{n^2}.$$

   Take $M_n = \frac{2}{n^2}$ and note that $\sum_{n=3}^{\infty} \frac{2}{n^2}$ converges.

Convergence of series of complex functions

Example. Show that the following the complex series is absolutely convergent when z is real but it becomes divergent when z is non-real.

$$\sum_{k=1}^{\infty} \frac{\sin kz}{k^2},$$

Solution (i) When z is real, we have

$$\left|\frac{\sin kz}{k^2}\right| \le \frac{1}{k^2}, \quad \text{for all positive integer values of } k.$$

Since $\sum_{k=1}^{\infty} 1/k^2$ is known to be convergent, then $\sum_{k=1}^{\infty} \sin kz/k^2$ is absolutely convergent for all z by virtue of the comparison test.

(ii) When z is non-real, we let $z = x + iy, y \neq 0$. From the relation

$$\frac{\sin kz}{k^2} = \frac{e^{-ky}e^{ikx} - e^{ky}e^{-ikx}}{2k^2 i},$$

we deduce that

$$\left|\frac{\sin kz}{k^2}\right| \ge \frac{e^{k|y|} - e^{-k|y|}}{2k^2} \to \infty \text{ as } k \to \infty.$$

Since $\left|\sin kz/k^2\right|$ is unbounded as $k \to \infty$, the series is divergent.

Convergence of series of complex functions

Example. Show that the geometric series $\sum_{n=0}^{\infty} z^n$ converges to $\frac{1}{1-z}$ on any closed subdisk $|z| \leq r < 1$ of the open unit disk $|z| < 1$.

Solution

To establish the convergence of the series for $|z| \leq r < 1$, we apply the *M*-test. We have

$$|f_n(z)| = |z^n| \leq r^n = M_n$$

for all $|z| \leq r$.

Since $\sum_{n=1}^{\infty} M_n = \sum_{n=0}^{\infty} r^n$ is convergent if $0 \leq r < 1$, we conclude that the series $\sum_{n=1}^{\infty} z_n$ converges for $|z| \leq r$.

Convergence of series of complex functions

❑ Some Stated Results

- For convergent infinite series of complex functions, properties such as continuity and analyticity of the continuous functions $f_k(z)$ are carried over to the sum $S(z)$.

- More precisely, suppose $f_k(z)$, $k = 1, 2, \cdots$, are all continuous (analytic) in the region of convergence, then the sum $S(z) = \sum f_k(z)$ is also continuous (analytic) in the same region.

- Further, a convergent infinite series allows for termwise differentiation and integration, that is,

$$\int_C \sum_{k=1}^{\infty} f_k(z)\, dz = \sum_{k=1}^{\infty} \int_C f_k(z)\, dz$$

$$\frac{d}{dz} \sum_{k=1}^{\infty} f_k(z) = \sum_{k=1}^{\infty} f_k'(z).$$

Power series

- We shall be particularly concerned with power series.
- A series of the form

$$a_0 + \sum_1^\infty a_n (z - z_0)^n = \sum_0^\infty a_n (z - z_0)^n$$

where z_0 and the a_n are complex constants, and z is any number (variable) in a stated region, is called a power series around the point z_0.

- Given a power series there exists a non-negative real number R, R can be zero or infinity, such that the power series converges absolutely for $|z - z_0| < R$, and diverges for $|z - z_0| > R$.
- R is called the ***radius of convergence.***
- $|z - z_0| = R$ is called the ***circle of convergence***.
- The radius of convergence R is given by $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$, as a consequence of the ***Ratio test***, given that the limit exists.
- The geometric series $\sum_{n=0}^{\infty} z^n$ is in fact a power series with $z_0 = 0$, $a_n = 1$,

  converges absolutely on $|z| < 1$ to the analytic function $1/(1 - z)$. Thus, its radius of convergence is $R = 1$.

Power series

(i) $R = \infty$ when $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$. The series converges in the whole plane.

(ii) $R = 0$ when $\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 0$. The series does not converges for any z other than z_0.

- Example

Find the circle of convergence for the following power series:

$$\sum_{k=1}^{\infty} \frac{1}{k}(z-i)^k,$$

Solution By the ratio test, we have

$$R = \lim_{k\to\infty} \frac{1/k}{1/(k+1)} = 1;$$

so the circle of convergence is $|z - i| = 1$.

Power series

- **Example.** Find the circle of convergence for the power series: $\sum_{k=1}^{\infty}\left(\frac{z}{k}\right)^k$

  Solution By the ratio test, we have

$$R=\lim_{k\to\infty}\frac{\left(\frac{1}{k}\right)^k}{\left(\frac{1}{k+1}\right)^{k+1}}=\lim_{k\to\infty}(k+1)\left(1+\frac{1}{k}\right)^k=\infty;$$

so the circle of convergence is the whole complex plane.

- The radius of convergence can also be found by $R=\frac{1}{\overline{\lim}_{n\to\infty}\sqrt[n]{|a_n|}}$, which is a consequence of the ***Root Test***.

- **Example.** Find the circle of convergence for the power series: $\sum_{k=1}^{\infty}\left(1+\frac{1}{k}\right)^{k^2}z^k.$

  Solution By the root test, we have

$$\frac{1}{R}=\lim_{k\to\infty}\sqrt[k]{\left(1+\frac{1}{k}\right)^{k^2}}=\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k=e,$$

so the circle of convergence is $|z|=1/e$.

Power series

- If a power series $\sum_{n=0}^{\infty} a_n z^n$ converges for some $z_0 \in \mathbb{C}$ then it converges for all $z \in \mathbb{C}$ such that $|z| < |z_0|$.

Proof. It follows from the hypothesis that there exist $M \geq 0$ such that $|a_n z_0^n| \leq M$ for all $n \in \mathbb{N}$.

Note that

$$|a_n z^n| = |a_n z_0|^n \left|\frac{z}{z_0}\right|^n \leq M \left|\frac{z}{z_0}\right|^n .$$

The proof now follows from the comparison test, and behavior of geometric series.

- If a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ converges at $z = z_1 (\neq z_0)$ then it converges (absolutely) for all $z \in \mathbb{C}$ such that $|z - z_0| < |z_1 - z_0|$.

- If it diverges at some point $z = z_1$, then it diverges at all points z that satisfy the inequality $|z - z_0| > |z_1 - z_0|$.

Power series

Some useful results

Let $S(z) = \sum \ a_n z^n$ over some circle of convergence C_1. Thus S is the function defined by the convergent power series. Then

(1) Function $S(z)$ is continuous at each z interior to C_1.

(2) Function $S(z)$ is analytic at each z interior to C_1.

(3) If C is any contour interior to C_1, then the power series can be integrated term by term, i.e.

$$\int_C S(z)\,dz = \sum_0^\infty a_n \int_C z^n\,dz.$$

(4) The power series can be differentiated term by term.

Thus for each z inside C_1,

$$S'(z) = \sum_1^\infty n a_n z^{n-1}.$$

Power series

Example. We have

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

By differentiating the series term-by term, it follows that

$$1 + 2z + 3z^2 + \cdots = \sum_{j=0}^{\infty} (j+1) z^j = \frac{1}{(1-z)^2}.$$

Its radius of convergence is 1.

Example. We have

$$1 - z + z^2 - z^3 - \cdots = (1+z)^{-1} = \frac{d}{dz} \mathrm{Log}\,(1+z)$$

By integrating the series term-by term, it follows that

$$\mathrm{Log}\,(1+z) = \sum_{j=0}^{\infty} (-1)^j \frac{z^{j+1}}{j+1} = z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots.$$

Its radius of convergence is also 1.

Power series

Suppose a power series represents the function $f(z)$ inside the circle of convergence, that is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

It is known that a power series can be differentiated termwise so that

$$\begin{aligned} f'(z) &= \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1} \\ f''(z) &= \sum_{n=2}^{\infty} n(n-1) a_n (z - z_0)^{n-2}, \cdots . \end{aligned}$$

Putting $z = z_0$ successively, we obtain

$$\begin{aligned} a_n &= \frac{f^{(n)}(z_0)}{n!}, \quad n = 0, 1, 2, \\ f(z) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n. \end{aligned}$$

Taylor's Series

- A power series represents an analytic function inside its circle of convergence.
- Can we expand an analytic function in Taylor series and how is the domain of analyticity related to the circle of convergence?

Taylor series theorem

Let f be analytic everywhere inside the circle $C : |z - z_0| = r_0$. Then, f has the series representation (called the ***Taylor series*** of f at z_0.)

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z-z_0)^j,$$

which converges for all z such that $|z - z_0| < r_0$.

- Taylor's series with $z_0 = 0$ is called the ***Maclaurin series*** of f.

- **Uniqueness of Taylor Series:** If

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

inside the circle of convergence, then the series is the Taylor series of f at z_0.

Taylor's Series

Example. Consider the function $\frac{1}{1-z}$, the Taylor series at $z = 0$ is given by

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

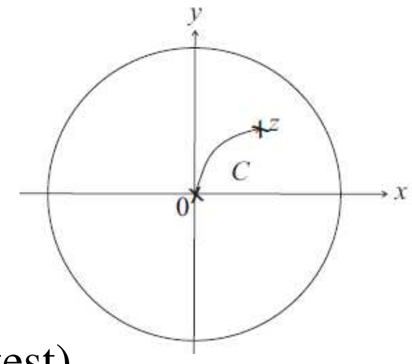
- The function has a singularity (i.e. the point at which the function is not analytic) at $z = 1$. The maximum distance from $z = 0$ to the nearest singularity is one, so the radius of convergence is one.
- Alternatively, the radius of convergence can be found by the ratio test, where

$$R = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = 1.$$

- If we integrate along the contour C inside the circle of convergence $|z| < 1$ from the origin to an arbitrary point z, we obtain

$$\int_C \frac{1}{1-\zeta} d\zeta = \sum_{n=0}^{\infty} \int_C \zeta^n \, d\zeta$$

$$-\text{Log}(1-z) = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$



The radius of convergence is again one (checked by the ratio test).

Taylor's Series

Example. Calculating the derivatives of all orders at $z_0 = 0$ of the entire functions $e^z, \cos z, \sin z, \cosh z, \sinh z$, we obtain the following Maclaurin series expansions, which are valid on $|z| < \infty$:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{z^j}{j!},$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j}}{(2j)!},$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots = \sum_{j=0}^{\infty} (-1)^j \frac{z^{2j+1}}{(2j+1)!},$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots = \sum_{j=0}^{\infty} \frac{z^{2j}}{(2j)!},$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots = \sum_{j=0}^{\infty} \frac{z^{2j+1}}{(2j+1)!}.$$

Taylor's Series

Example. The consecutive derivatives of Log z are $1/z, -1/z^2, 2/z^3 \cdots$; in general,

$$\frac{d^j \text{Log}\, z}{dz^j} = (-1)^{j+1}(j-1)!z^{-j}, \quad j = 1, 2, \cdots.$$

Evaluating these at $z = 1$, we find the Taylor series expansion

$$\begin{aligned} \text{Log}\, z &= 0 + (z-1) - \frac{(z-1)^2}{2!} + 2!\frac{(z-1)^3}{3!} - 3!\frac{(z-1)^4}{4!} + \cdots \\ &= \sum_{j=1}^{\infty} \frac{(-1)^{j+1}(z-1)^j}{j}. \end{aligned}$$

This is valid for $|z - 1| < 1$, the largest open disk centered at 1 over which Log z is analytic.

Example. Write $1/(9 + z^2) = 1/[9(1 - w)]$, where $w = -z^2/9$. Thus, as long as $|w| < 1$; i.e., $|-z^2/9| < 1$ or $|z| < 3$, we have

$$\frac{1}{9+z^2} = \frac{1}{9}\sum_{j=0}^{\infty} w^j = \frac{1}{9}\sum_{j=0}^{\infty}\left(-\frac{z^2}{9}\right)^n.$$

Taylor's Series

Example. Use partial fractions to obtain

$$\begin{aligned}
\frac{1}{z^2-5z+6} &= \frac{1}{(z-3)(z-2)} = \frac{1}{z-3} - \frac{1}{z-2} \\
&= \frac{1}{2}\left(\frac{1}{1-z/2}\right) - \frac{1}{3}\left(\frac{1}{1-z/3}\right) \\
&= \frac{1}{2}\sum_{j=0}^{\infty}\left(\frac{z}{2}\right)^j - \frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{z}{3}\right)^j \\
&= \sum_{j=0}^{\infty}\left[\left(\frac{1}{2}\right)^{j+1} - \left(\frac{1}{3}\right)^{j+1}\right] z^j, \quad |z| < 2.
\end{aligned}$$

Taylor's Series

Example. Consider the function $f(z) = (1 + 2z)/(z^3 + z^4)$. We cannot find a Maclaurin series for $f(z)$ since it is not analytic at $z = 0$. However, we can write $f(z)$ as

$$f(z) = \frac{1}{z^3}\left(2 - \frac{1}{1+z}\right)$$

Now $1/(1 + z)$ has a Taylor series expansion around the point $z = 0$.

Thus, when $0 < |z| < 1$, it follows that

$$\begin{aligned} f(z) &= \frac{1}{z^3}(2 - 1 + z - z^2 + z^3 - z^4 + z^5 - z^6 + \cdots) \\ &= \frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z} + 1 - z + z^2 - z^3 + \cdots. \end{aligned}$$

- This is not a Maclaurin series: the first two terms are unexpected, and function f has a singularity at $z = 0$.

Question Perhaps there are other interesting series to investigate?

Laurent series

- Consider an infinite power series with negative power terms

$$\sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

how to find the region of convergence? Set $w = \dfrac{1}{z - z_0}$, the series becomes $\sum_{n=1}^{\infty} b_n w^n$, a Taylor series in w. Suppose $R' = \lim_{n \to \infty} \left| \dfrac{b_n}{b_{n+1}} \right|$ exists, then $\sum_{n=1}^{\infty} b_n w^n$ converges for $|w| < R' \Leftrightarrow |z - z_0| > \dfrac{1}{R'}$.

- *Special cases*: (i) $R' = 0$ and (ii) $\dfrac{1}{R'} = 0$.

  (i) In the first case: the series does not converge for any z, not even at $z = z_0$.

  (ii) In the second case: by virtue of the ratio test, the region of convergence is the whole complex plane except at $z = z_0$, that is, $|z - z_0| > 0$.

Laurent series

For the more general case (Laurent series at z_0)

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n + \underbrace{\sum_{n=1}^{\infty} b_n(z-z_0)^{-n}}_{\text{principal part}},$$

suppose $R = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|$ and $R' = \lim_{n\to\infty}\left|\frac{b_n}{b_{n+1}}\right|$ exists, and $RR' > 1$,

then inside the annular domain

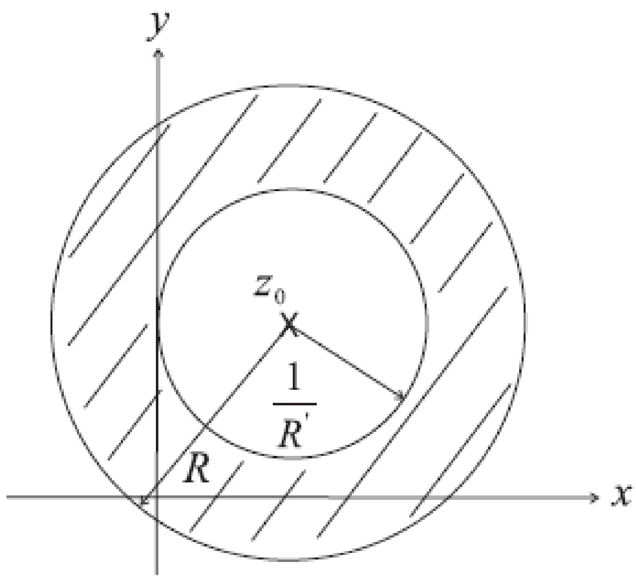
$$\left\{z : \quad \frac{1}{R'} < |z-z_0| < R\right\}$$

the Laurent series is convergent.

The annulus degenerates into

(i) hollow plane if $R = \infty$

(ii) punctured disc if $R' = \infty$.



Laurent series

- **Laurent series theorem**

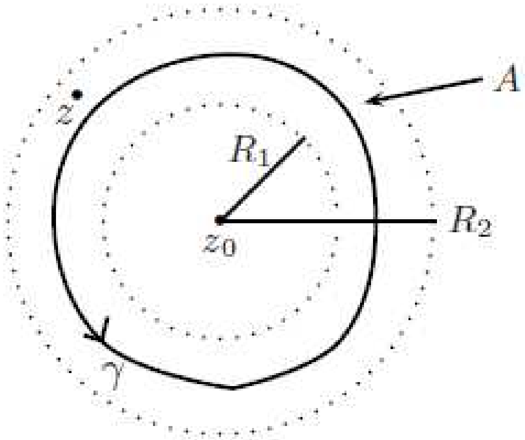
Let $f(z)$ be analytic in the annulus $A : R_1 < |z - z_0| < R_2$, then $f(z)$ can be represented by the Laurent series,

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z - z_0)^k,$$

which converges to $f(z)$ throughout the annulus. The Laurent coefficients are given by

$$c_k = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta, \quad k = 0, \pm 1, \pm 2, \cdots,$$

where C is any simple closed contour lying completely inside the annulus and going around the point z_0.



Laurent series

Remarks

1) A Laurent series defines a function f(z) in its annular region of convergence. The Laurent series theorem states that a function analytic in an annulus can be expanded in a Laurent series expansion.

2) Suppose f(z) is analytic in the full disc: $|z - z_0| < R_2$ (without the punctured hole), then the integrand in calculating c_k for negative k becomes analytic in $|z - z_0| < R_2$. Hence, $c_k = 0$ for $k = -1, -2, \cdots$.
The Laurent series is reduced to a Taylor series.

3) When k = −1,

$$c_{-1} = \frac{1}{2\pi i} \oint_C f(\zeta)\, d\zeta.$$

We may find c_{-1} by any means, so a contour integral can be evaluated without resort to direct integration.

Laurent series

Example. From the Maclaurin series expansion of e^z, it follows that the Laurent series expansion of $e^{1/z}$ about 0 is

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! z^n}.$$

The function $e^{1/z}$ is analytic everywhere except at $z = 0$ so that the annulus of convergence is $|z| > 0$. We observe

$$\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \frac{1/(n+1)!}{1/n!} = 0 \quad \text{so that } \frac{1}{R'} = 0.$$

Lastly, we consider $\oint_C e^{1/z}\,dz$, where the contour C is $|z| = 1$. Since C lies completely inside the punctured disc $|z| > 0$, we have

$$\oint_C e^{1/z}\,dz = 2\pi i(\text{coefficient of } \frac{1}{z} \text{ in Laurent expansion}) = 2\pi i$$

- By comparing the coefficients, we obtain

$$\oint_C \frac{e^{1/z}}{z^{1-n}}\,dz = \frac{2\pi i}{n!}, \qquad n \in \mathbb{N}.$$

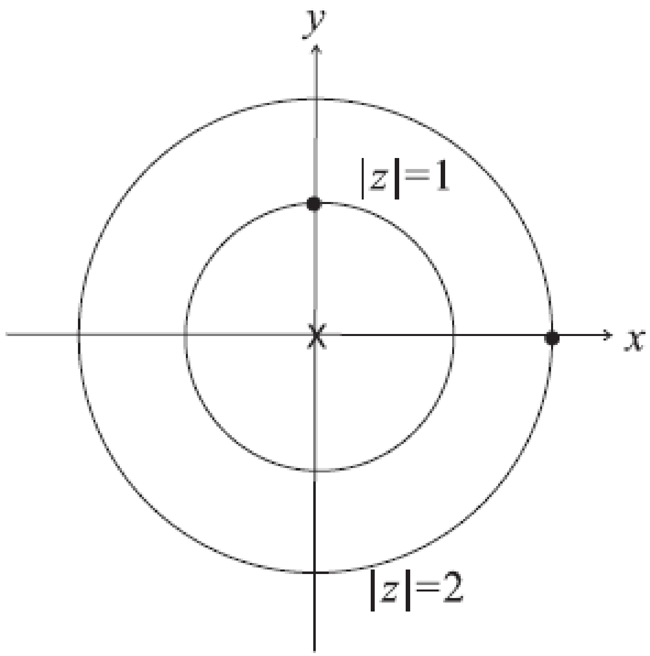
Laurent series

Example

Find all the possible Taylor and Laurent series expansions of

$$f(z) = \frac{1}{(z-i)(z-2)} \quad \text{at } z_0 = 0.$$

Specify the region of convergence (solid disc or annulus) of each of the above series.



Laurent series

Solution

There are two isolated singularities, namely, at $z = i$ and $z = 2$. The possible circular or annular regions of analyticity are

(i) $|z| < 1$, (ii) $1 < |z| < 2$, (iii) $|z| > 2$.

(i) For $|z| < 1$

$$\frac{1}{z-i} = \frac{i}{1-\frac{z}{i}} = i\left[1+\frac{z}{i}+\cdots+\left(\frac{z^n}{i^n}\right)+\right] \quad \text{for} \quad |z|<1$$

$$\frac{1}{z-2} = \left(-\frac{1}{2}\right)\frac{1}{1-\frac{z}{2}} = \left(-\frac{1}{2}\right)\left[1+\frac{z}{2}+\cdots+\left(\frac{z}{2}\right)^n+\cdots\right] \quad \text{for} \quad |z|<2$$

$$f(z) = \frac{1}{i-2}\left(\frac{1}{z-i}-\frac{1}{z-2}\right) = \frac{1}{i-2}\left[i\sum_{n=0}^{\infty}\left(\frac{z}{i}\right)^n+\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n\right]$$

$$= \frac{1}{i-2}\sum_{n=0}^{\infty}\left[\left(\frac{1}{i}\right)^{n-1}+\frac{1}{2^{n+1}}\right]z^n.$$

This is a Taylor series which converges inside the solid disc $|z| < 1$.

Laurent series

(ii) For $1 < |z| < 2$

$$\frac{1}{z-i} = \frac{1}{z}\frac{1}{1-\frac{i}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{i}{z}\right)^n \qquad \text{valid for } |z| > 1$$

$$\frac{1}{z-2} = -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \qquad \text{valid for } |z| < 2$$

$$f(z) = \frac{1}{i-2}\left[\sum_{n=0}^{\infty}\left(\frac{i^n}{z^{n+1}} + \frac{z^n}{2^{n+1}}\right)\right] \qquad \text{valid for } 1 < |z| < 2.$$

(iii) For $|z| > 2$

$$\frac{1}{z-i} = \sum_{n=0}^{\infty}\frac{i^n}{z^{n+1}} \qquad \text{valid for } |z| > 1$$

$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} \qquad \text{valid for } |z| > 2$$

$$f(z) = \frac{1}{i-2}\left[\sum_{n=0}^{\infty}(i^n - 2^n)\frac{1}{z^{n+1}}\right] \qquad \text{valid for } |z| > 2.$$

Laurent series

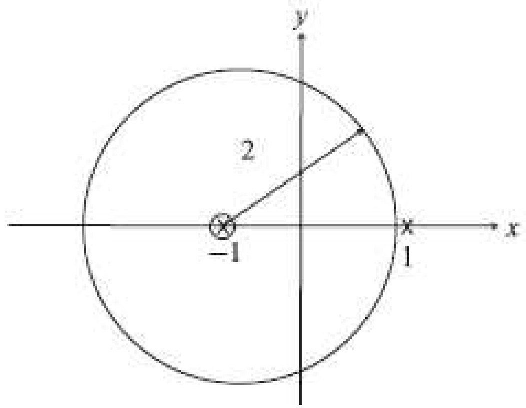
Example

Find all possible Laurent series of

$$f(z) = \frac{1}{1 - z^2} \quad \text{at the point } \alpha = -1.$$

Solution

The function has two isolated singular points at $z = 1$ and $z = -1$. There exist two annular regions (i) $0 < |z + 1| < 2$ and (ii) $|z + 1| > 2$ where the function is analytic everywhere inside the respective region.



Laurent series

(i) $0 < |z+1| < 2$

$$\begin{aligned}\frac{1}{1-z^2} &= \frac{1}{2(z+1)\left(1-\frac{z+1}{2}\right)} \\ &= \frac{1}{2(z+1)}\sum_{k=0}^{\infty}\frac{(z+1)^k}{2^k} \\ &= \frac{1}{2(z+1)} + \frac{1}{4} + \frac{1}{8}(z+1) + \frac{1}{16}(z+1)^2 + \cdots .\end{aligned}$$

The above expansion is valid provided $\left|\frac{z+1}{2}\right| < 1$ and $z+1 \neq 0$.

Given that $0 < |z+1| < 2$, the above requirement is satisfied.

For any simple closed curve C_1 lying completely inside $|z+1| < 2$ and encircling the point $z = -1$, we have

$$c_{-1} = \frac{1}{2\pi i}\oint_{C_1}\frac{1}{1-z^2}\,dz = \frac{1}{2}.$$

Laurent series

(ii) $|z+1| > 2$

$$\begin{aligned}\frac{1}{1-z^2} &= -\frac{1}{(z+1)^2\left(1-\frac{2}{z+1}\right)} \\ &= -\frac{1}{(z+1)^2}\sum_{k=0}^{\infty}\frac{2^k}{(z+1)^k} \text{ since } \left|\frac{2}{z+1}\right| < 1 \\ &= -\left[\frac{1}{(z+1)^2}+\frac{2}{(z+1)^3}+\frac{4}{(z+1)^4}+\cdots\right].\end{aligned}$$

For any simple closed curve C_2 encircling the circle: $|z+1| = 2$, we have

$$c_1 = \frac{1}{2\pi i}\oint_{C_2}\frac{1}{1-z^2}\,dz = 0.$$

Thank you.

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