Lecture 8
Infinite Sequences and Series: Part I

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Sequences

- A ***sequence*** is a list of numbers written in a definite order:

$$a_1, a_2, \ldots, a_n, a_{n+1}, \ldots$$

$$\begin{gathered} a_1 - \text{first term} \\ a_2 - \text{second term} \\ \vdots \\ a_n - n^{th} \text{ term} \\ a_{n+1} - (n+1)^{\text{st}} \text{ term} \\ \vdots \end{gathered}$$

- Sequences are written in a few different ways, all equivalent

$$\{a_1, a_2, \ldots, a_n, a_{n+1}, \ldots\} \qquad \{a_n\} \qquad \{a_n\}_{n=1}^{\infty}$$

- It sometimes helps to think of a sequence as a function that assigns, to each natural number n (index), the number a_n (value).

Sequences

the sequence	first few numbers in the sequence
$\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$	$\underbrace{2}_{n=1}, \underbrace{\frac{3}{4}}_{n=2}, \underbrace{\frac{4}{9}}_{n=3}, \underbrace{\frac{5}{16}}_{n=4}, \underbrace{\frac{6}{25}}_{n=5}, \ldots$
$\{n\}_{n=1}^{\infty}$	$1, 2, 3, 4, \ldots$
$\left\{\frac{(-1)^{n+1}}{2^n}\right\}_{n=0}^{\infty}$	$-1, \frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, -\frac{1}{16}, \ldots$
$\{0\}_{n=1}^{\infty}$	$0, 0, 0, 0, \ldots$
$\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$	$\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$
$\left\{\left(-\frac{1}{3}\right)^n\right\}_{n=1}^{\infty}$	$-\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \frac{1}{81}, \ldots$
$\{a_n\}_{n=1}^{\infty}, a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$	$1, 2, 2\frac{1}{2}, 2\frac{2}{3}, 2\frac{17}{24}, 2\frac{43}{60}, \ldots$
$\{b_n\}_{n=1}^{\infty}, b_n = x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$	$x, x - \frac{x^3}{3!}, x - \frac{x^3}{3!} + \frac{x^5}{5!}, \ldots$

Table 5.1: **Examples of sequences**

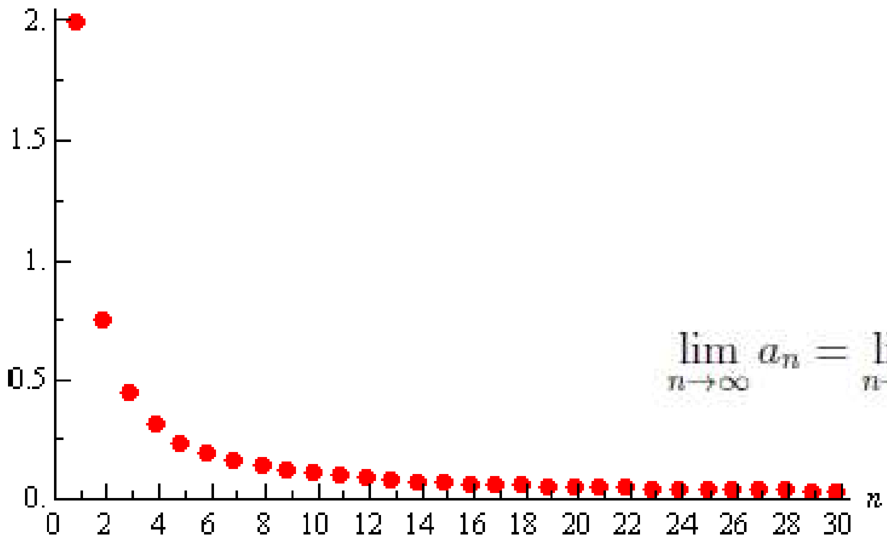
Sequences

- To graph the sequence $\{a_n\}$ we plot the points (n, a_n) as n ranges over all possible values on a graph. For instance, let's graph the sequence

$$\left\{\frac{n+1}{n^2}\right\}_{n=1}^{\infty}$$

  - The first few points on the graph are,

$$(1,2),\ \left(2,\frac{3}{4}\right),\ \left(3,\frac{4}{9}\right),\ \left(4,\frac{5}{16}\right),\ \left(5,\frac{6}{25}\right),\ \ldots$$



$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n^2} = 0$$

Sequences

Working Definition of Limit. We say that

$$\lim_{n\to\infty} a_n = L$$

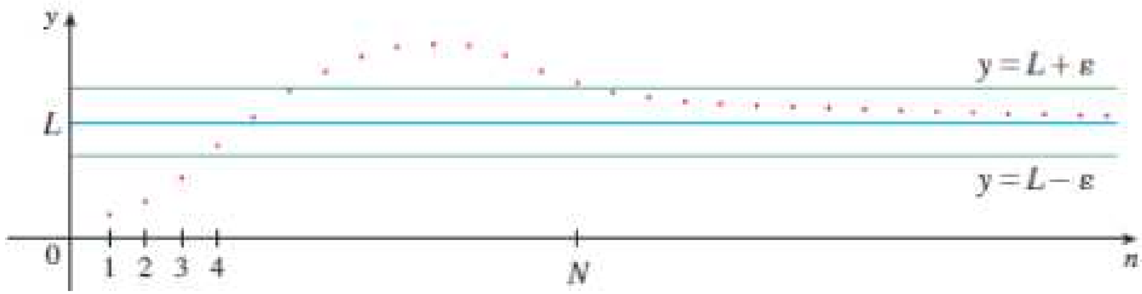
if we can make a_n as close to L as we want for all sufficiently large n. In other words, the value of the a_n's approach L as n approaches infinity.

Precise Definition of Limit. We say that

$$\lim_{n\to\infty} a_n = L$$

if for every number $\varepsilon > 0$ there is an integer N such that

$$|a_n - L| < \varepsilon \qquad \text{whenever} \qquad n > N$$



Sequences

Example 73 *Show that*

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

Solution. To prove this let $\varepsilon > 0$ be given. We have to find an N such that

$$|a_n| < \varepsilon \text{ for all } n > N.$$

The a_n are all positive, so $|a_n| = a_n$, and hence

$$|a_n| < \varepsilon \iff \frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon},$$

which prompts us to choose $N = 1/\varepsilon$. The calculation we just did shows that if $n > \frac{1}{\varepsilon} = N$, then $|a_n| < \varepsilon$. That means that $\lim_{n\to\infty} a_n = 0$.

Sequences

- We say that

$$\lim_{n\to\infty} a_n = \infty$$

  if we can make a_n as large as we want for all sufficiently large *n*.

- Similarly, we say that

$$\lim_{n\to\infty} a_n = -\infty$$

  if we can makea_n as large and negative as we want for all sufficiently large *n*.

- If $\lim_{n\to\infty} a_n$ an exists and is finite we say that the sequence ***converges*** (or is ***convergent***). Otherwise, we say the sequence ***diverges*** (or is ***divergent***).

- Most limits of most sequences can be found using one of the following theorems.

Theorem 74 *Given the sequence* $\{a_n\}$ *if we have a function* $f(x)$ *such that* $f(n) = a_n$ *and* $\lim_{x\to\infty} f(x) = L$ *then* $\lim_{n\to\infty} a_n = L$.

Sequences

Example. Determine if the following sequence converge or diverge. If the sequence converges determine its limit.

$$\left\{\frac{e^{2n}}{n}\right\}_{n=1}^{\infty}$$

Solution. Let's define

$$f(x) = \frac{e^{2x}}{x}$$

and note that,

So, the sequence in this part diverges (to ∞).

$$f(n) = \frac{e^{2n}}{n}$$

Theorem 74 says that all we need to do is take the limit of the function.

$$\lim_{n\to\infty} \frac{e^{2n}}{n} = \lim_{x\to\infty} \frac{e^{2x}}{x} = \lim_{x\to\infty} \frac{2e^{2x}}{1} = \infty$$

So, the sequence in this part diverges (to ∞).

Sequences

Properties. If $\{a_n\}$ and $\{b_n\}$ are both convergent sequences then,

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$

2. $\lim_{n\to\infty} c a_n = c \lim_{n\to\infty} a_n$

3. $\lim_{n\to\infty}(a_n\, b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$

4. $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, provided $\lim_{n\to\infty} b_n \neq 0$

5. $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$ provided $a_n \geq 0$

Sequences

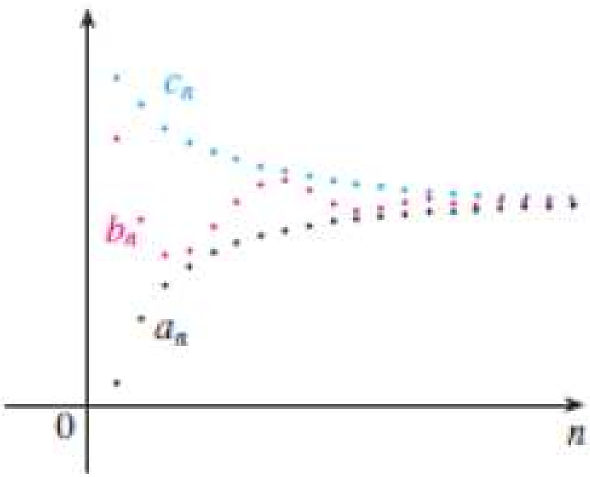


Figure 5.2: The sequence $\{b_n\}$ is squeezed between the sequences $\{a_n\}$ and $\{c_n\}$.

Theorem 76 (Squeeze Theorem for Sequences) *If* $a_n \leq b_n \leq c_n$ *for all* $n > N$ *for some* N *and* $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ *then* $\lim_{n\to\infty} b_n = L$.

Squeeze Theorem for Sequences is also called ***Sandwich Theorem***.

Sequences

Theorem 77 *If* $\lim_{n\to\infty} |a_n| = 0$ *then* $\lim_{n\to\infty} a_n = 0$.

Example. Determine if the following sequences converge or diverge. If the sequence converges determine its limit.

$$\left\{ \frac{(-1)^n}{n} \right\}_{n=1}^{\infty}$$

Solution.

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore, since the limit of the sequence terms with absolute value bars on them goes to zero we know by Theorem 77 that,

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

which also means that the sequence converges to a value of zero.

- Note that in order for Theorem 77 to hold the limit MUST be zero and it won't work for a sequence whose limit is not zero. For example, consider the sequence

$$\{(-1)^n\}_{n=0}^{\infty} = \{1, -1, 1, -1, 1, -1, 1, -1, \ldots\}_{n=0}^{\infty}$$

Sequences

Theorem 79 *The sequence* $\{r^n\}_{n=0}^{\infty}$ *converges if* $-1 < r \leq 1$ *and diverges for all other values of* r. *Also,*

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$

Theorem 80 *If* $f(x)$ *is a function which is continuous at* $x = A$, *and* a_n *is a sequence which converges to* A, *then*

$$\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(A).$$

Example: Since $\lim_{n\to\infty} 1/n = 0$ and since $f(x) = \cos x$ is continuous at $x = 0$ we have

$$\lim_{n\to\infty} \cos\frac{1}{n} = \cos\left(\lim_{n\to\infty}\frac{1}{n}\right) = \cos 0 = 1.$$

Sequences

- **Limits of rational functions**

You can compute the limit of any rational function of n by dividing numerator and denominator by the highest occurring power of n. Here is an example:

$$\lim_{n\to\infty} \frac{2n^2 - 1}{n^2 + 3n} = \lim_{n\to\infty} \frac{2 - \left(\frac{1}{n}\right)^2}{1 + 3 \cdot \frac{1}{n}} = \frac{2 - 0^2}{1 + 3 \cdot 0^2} = 2.$$

- **Example. Application of the Sandwich theorem.**

$$\lim_{n\to\infty} \frac{1}{\sqrt{n^2+1}} = 0$$

Since $\sqrt{n^2 + 1} > \sqrt{n^2} = n$ we have

$$0 < \frac{1}{\sqrt{n^2 + 1}} < \frac{1}{n}.$$

The sequences "0" and $\frac{1}{n}$ both go to zero, so the Sandwich theorem implies that $1/\sqrt{n^2 + 1}$ also goes to zero.

Sequences

Example: factorial beats any exponential

In this example we'll show that

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \text{ for any real number } x.$$

If $|x| \le 1$ then this is easy, for we would have $|x^n| \le 1$ for all $n \ge 0$ and thus

$$\left|\frac{x^n}{n!}\right| \le \frac{1}{n!} = \frac{1}{1\cdot \underbrace{2\cdot 3\cdots(n-1)\cdot n}_{n-1 \text{ factors}}} \le \frac{1}{1\cdot \underbrace{2\cdot 2\cdots 2\cdot 2}_{n-1 \text{ factors}}} = \frac{1}{2^{n-1}}.$$

So the Sandwich Theorem applies and tells us that it is true, at least when $|x| \le 1$.

If $|x| > 1$ then we need a slightly longer argument. For arbitrary x we first choose an integer $N \ge 2x$. Then for all $n \ge N$ one has

$$\frac{x^n}{n!} \le \frac{|x|\cdot|x|\cdots|x|\cdot|x|}{1\cdot 2\cdot 3\cdots n} \le \frac{N\cdot N\cdot N\cdots N\cdot N}{1\cdot 2\cdot 3\cdots n}\left(\frac{1}{2}\right)^n \qquad \text{use } |x| \le \frac{N}{2}$$

$$\underbrace{\frac{N}{1}\cdot\frac{N}{2}\cdot\frac{N}{3}\cdots\frac{N}{N}}_{=N^N/N!}\cdot\frac{N}{N+1}\cdots\frac{N}{n} = \frac{N^N}{N!}\cdot\underbrace{\frac{N}{N+1}}_{<1}\cdot\underbrace{\frac{N}{N+2}}_{<1}\cdots\underbrace{\frac{N}{n}}_{<1} \le \frac{N^N}{N!}$$

Hence we have, if $2|x| \le N$ and $n \ge N$,

$$\left|\frac{x^n}{n!}\right| \le \frac{N^N}{N!}\left(\frac{1}{2}\right)^n$$

the whole thing to converge to zero as $n \to \infty$.

Series - Basics

- A ***series*** is an infinite sum:

$$a_1 + a_2 + a_3 + \cdots = \sum_{k=1}^{\infty} a_k.$$

- The result of adding the first *n* terms,

$$s_n = a_1 + a_2 + \cdots + a_n$$

  is called the n^{th} ***partial sum***.

- The partial sums form themselves a new sequence, $\{s_n\}_{n=1}^{\infty}$:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + a_4 + \cdots + a_n = \sum_{i=1}^{n} a_i \end{aligned}$$

Series - Basics

- We say the ***series converges*** to a number S, *which we call the sum of the series*, if $\lim_{n\to\infty} s_n$ exists, and if

$$S = \lim_{n\to\infty} s_n$$
$$= \lim_{n\to\infty} a_1 + a_2 + \cdots + a_n.$$

- If the limit does not exist, then we say the ***series diverges***. If the limit does exist, then we write either

$$S = \sum_{k=1}^{\infty} a_k.$$
$$= a_1 + a_2 + a_3 + \cdots$$

- **Example**. Determine if the following series converge or diver $\sum_{n=1}^{\infty} n$ ..

Solution.

$$s_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$$

Therefore, the sequence of partial sums diverges to ∞ and so the series also diverges.

Series - Basics

- **Example**. Determine if the following series converge or diverge $\sum_{n=0}^{\infty} (-1)^n$.

Solution.

$$s_0 = 1$$
$$s_1 = 1 - 1 = 0$$
$$s_2 = 1 - 1 + 1 = 1$$
$$s_3 = 1 - 1 + 1 - 1 = 0$$
$$etc.$$

So, it looks like the sequence of partial sums is,

$$\{s_n\}_{n=0}^{\infty} = \{1, 0, 1, 0, 1, 0, 1, 0, 1, \ldots\}$$

and this sequence diverges since $\lim_{n\to\infty} s_n$ doesn't exist. Therefore, the series also diverges

The geometric series

A ***geometric series*** is any series that can be written in the form,

$$\sum_{n=1}^{\infty} ar^{n-1} \quad \text{or} \quad \sum_{n=0}^{\infty} ar^{n}$$

If we start with the first form it can be shown that the partial sums are,

$$s_n = \frac{a\left(1-r^n\right)}{1-r}$$

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}$$

provided $-1 < r < 1$

The ***geometric series*** will converge if $-1 < r < 1$, which is usually written $|r| < 1$, its value is,

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^{n} = \frac{a}{1-r}$$

If $|r| \geq 1$, the geometric series is divergent.

The geometric series

For example, the geometric sum formula tells us that

$$1+\left(\tfrac{1}{2}\right)+\left(\tfrac{1}{2}\right)^2+\left(\tfrac{1}{2}\right)^3+\cdots+\left(\tfrac{1}{2}\right)^n=\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}=2\left(1-\frac{1}{2^{n+1}}\right)=2-\frac{1}{2^n}.$$

Since $\lim_{n\to\infty}\frac{1}{2^n}=0$, we find

$$1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\cdots=2.$$

Example. Determine if the following series converge or diverge. If it converges give the value of the series.

$$\sum_{n=1}^{\infty}9^{-n+2}4^{n+1}$$

Solution.

$$\sum_{n=1}^{\infty}9^{-n+2}4^{n+1}=\sum_{n=1}^{\infty}9^{-(n-2)}4^{n+1}=\sum_{n=1}^{\infty}\frac{4^{n+1}}{9^{n-2}}$$

$$=\sum_{n=1}^{\infty}\frac{4^{n-1}4^2}{9^{n-1}9^{-1}}=\sum_{n=1}^{\infty}16\,(9)\,\frac{4^{n-1}}{9^{n-1}}$$

$$=\sum_{n=1}^{\infty}144\left(\frac{4}{9}\right)^{n-1}=\frac{144}{1-\frac{4}{9}}=\frac{9}{5}(144)=\frac{1296}{5}$$

The geometric series

Consider the following series written in two separate ways

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \sum_{n=3}^{\infty} a_n$$

- **Fact**. $\sum_{n=3}^{\infty} a_n$ is a convergent series iff $\sum_{n=0}^{\infty} a_n$ is a convergent series.

For example the series $\sum_{n=3}^{\infty} 9^{-n+2}4^{n+1}$ is convergent since

$$\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} = 9^1 4^2 + 9^0 4^3 + \sum_{n=3}^{\infty} 9^{-n+2}4^{n+1} = 208 + \sum_{n=3}^{\infty} 9^{-n+2}4^{n+1}$$

and we know from the previous example that the series $\sum_{n=1}^{\infty} 9^{-n+2}4^{n+1}$ is convergent. Moreover, we can now use the value of the series from the previous example to get the value of this series.

$$\sum_{n=3}^{\infty} 9^{-n+2}4^{n+1} = \sum_{n=1}^{\infty} 9^{-n+2}4^{n+1} - 208 = \frac{1296}{5} - 208 = \frac{256}{5}$$

Telescoping series

It turns out that

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}+\cdots=1.$$

There is a trick that allows us to compute the partial sum. The trick begins with the miraculous observation that

$$\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}.$$

$$\begin{aligned} s_n &= \frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)} \\ &= (\frac{1}{1}-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\cdots+(\frac{1}{n}-\frac{1}{n+1}) \\ &= \frac{1}{1}\underbrace{-\frac{1}{2}+\frac{1}{2}}_{=0}\underbrace{-\frac{1}{3}+\frac{1}{3}}_{=0}+\cdots\underbrace{-\frac{1}{n}+\frac{1}{n}}_{=0}-\frac{1}{n+1} \\ &= 1-\frac{1}{n+1}. \end{aligned}$$

Once we have the formula for s_n it is easy to compute the sum:

$$S=\lim_{n\to\infty}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{n(n+1)}=\lim_{n\to\infty}1-\frac{1}{n+1}=1.$$

Telescoping series

- **Example**. Determine if the following series converges or diverges. If it converges find its value.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+4n+3}$$

Solution. We first need the partial sums for this series: $s_n = \sum_{i=1}^{n} \frac{1}{i^2+4i+3}$

Now, let's notice that we can use partial fractions on the series term to get,

$$\frac{1}{i^2+4i+3} = \frac{1}{(i+1)(i+3)} = \frac{\frac{1}{2}}{i+1} - \frac{\frac{1}{2}}{i+3}$$

$$s_n = \sum_{i=1}^{n} \left(\frac{\frac{1}{2}}{i+1} - \frac{\frac{1}{2}}{i+3} \right) = \frac{1}{2} \sum_{i=1}^{n} \left(\frac{1}{i+1} - \frac{1}{i+3} \right)$$

$$= \frac{1}{2} [\left(\frac{1}{2} - \frac{1}{4} \right) + \left(\frac{1}{3} - \frac{1}{5} \right) + \left(\frac{1}{4} - \frac{1}{6} \right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2} \right) + \left(\frac{1}{n+1} - \frac{1}{n+3} \right)]$$

$$= \frac{1}{2} \left[\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3} \right]$$

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{2} \left(\frac{5}{6} - \frac{1}{n+2} - \frac{1}{n+3} \right) = \frac{5}{12}$$

So, this series is convergent and its value is 5/12.

Properties of convergent series

Theorem 84 *If* $\sum a_n$ *and* $\sum b_n$ *are both convergent series then,*

1. $\sum ca_n$*, where* c *is any number, is also convergent and*

$$\sum ca_n = c\sum a_n$$

2. $\sum_{n=k}^{\infty} a_n \pm \sum_{n=k}^{\infty} b_n$ *is also convergent and,*

$$\sum_{n=k}^{\infty} a_n \pm \sum_{n=k}^{\infty} b_n = \sum_{n=k}^{\infty} (a_n \pm b_n)$$

Example.

$$\sum_{n=1}^{\infty}\left(\frac{4}{n^2+4n+3} - 9^{-n+2}4^{n+1}\right) = \sum_{n=1}^{\infty}\frac{4}{n^2+4n+3} - \sum_{n=1}^{\infty}9^{-n+2}4^{n+1}$$

$$= 4\sum_{n=1}^{\infty}\frac{1}{n^2+4n+3} - \sum_{n=1}^{\infty}9^{-n+2}4^{n+1}$$

$$= 4\left(\frac{5}{12}\right) - \frac{1296}{5}$$

$$= -\frac{3863}{15}$$

Properties of convergent series

- The *harmonic series* takes the form $\sum_{n=1}^{\infty} \frac{1}{n}$

- The harmonic series is divergent and we'll need to wait until the next lecture to show that.

- By noting that

$$\sum_{n=1}^{\infty} \frac{5}{n} = 5 \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \sum_{n=4}^{\infty} \frac{1}{n} \qquad \Rightarrow \qquad \sum_{n=4}^{\infty} \frac{1}{n} = \left(\sum_{n=1}^{\infty} \frac{1}{n} \right) - \frac{11}{6}$$

  - the two series $\sum_{n=1}^{\infty} \frac{5}{n}$ and $\sum_{n=4}^{\infty} \frac{1}{n}$ are divergent.

- Multiplying a series by a constant will not change the convergence/divergence of the series.

- Adding or subtracting a constant from a series will not change the convergence /divergence of the series.

Divergence Test

Theorem 87 *If* $\sum a_n$ *converges then* $\lim_{n\to\infty} a_n = 0$.

Divergence Test. If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ will diverge.

Do NOT misuse this test.

Example. Determine if the following series is convergent or divergent.

$$\sum_{n=0}^{\infty} \frac{4n^2 - n^3}{10 + 2n^3}$$

Solution.

$$\lim_{n\to\infty} \frac{4n^2 - n^3}{10 + 2n^3} = -\frac{1}{2} \neq 0$$

The limit of the series terms isn't zero and so by the Divergence Test the series diverges.

Thank you.

Dr. Moataz El-Zekey