Sheet 4

1. The function $\varphi(F) = \int_a^b F$ is a linear function.
   (a) True (b) False

2. If $a \leq b$, then $\int_a^b |F| \leq \left|\int_a^b F\right|$.
   (a) True (b) False

3. A smooth continuous arc is a contour.
   (a) True (b) False

4. If C is a contour, then C must be a smooth continuous arc.
   (a) True (b) False

5. If C is the line segment from 0 to $1 + i$, then $\int_C 2z\,dz = 2i$.
   (a) True (b) False

6. If C is the line segment from 0 to 1, followed by the line segment form 1 to $1 + i$, then $\int_C 2z\,dz = 2i$.
   (a) True (b) False

7. If C is the upper half, from π to 0, of the circle $|z| = \sqrt{2}$, then $\int_C \bar{z}\,dz = 2\pi i$.
   (a) True (b) False

8. If C is the circle center O, radius k and fully traversed, then $\left|\int_C 1/z\,dz\right| \leq 2\pi$.
   (a) True (b) False

9. If $f(z) = z/(z^2 - 4)$, and C is the circle $|z - i| = 1$ traversed positively, then $\int_C f(z)\, dz =$

(a) $-2\pi i$ (b) 0 (c) $2\pi i$

10. If C is the (positive) contour $|z| = 4$, then $\int_C \frac{2z}{z-2} dz = 8\pi i$.

(a) True (b) False

11. If C is the (positive) contour $|z| = 1$, then $\int_C \frac{2z}{z-2} dz = 8\pi i$.

(a) True (b) False

12. If C is the (positive) contour $|z| = 4$, then $\int_C \frac{2z}{(z-2)^2} dz = 8\pi i$.

(a) True (b) False

13. If C is the (positive) contour $|z| = 1$, then $\int_C \frac{2z}{(z-2)^2} dz = 8\pi i$.

(a) True (b) False

14. If $f = u + iv$ is entire, then all partial derivatives of u, v are continuous everywhere.

(a) True (b) False

15. Evaluate $\int_0^{3+i} z^2\, dz$:

a) Along the line $y = x/3$

b) Along the real axis to 3 and then vertically to $3 + i$

c) Along the imaginary axis to i and then horizontally to $3 + i$

16. Evaluate $\int_0^{3+i} (\bar{z})^2\, dz$ along each of paths used in Exercise 15.

17. If C is the boundary of the square with vertices at the points $z = 0, z = 1, z = 1 + i$ and $z = i$, Show that $\int_C (3z + 1)\, dz = 0$.

18. Consider the integral

$$\frac{1}{2\pi i} \int_C \frac{e^{2z}}{z^2 + 1} dz$$

What is the value of this integral if the path of integration is the circle $|z| = 1/2$?

19. Evaluate each of these integrals where the path is an arbitrary contour between the points represented by the limits:

$(a) \int_{1+i}^{2+3i} (3z^2 + 2z + 1)\, dz$ $\quad$ $(b) \int_{1}^{3+i} \sin 2z\, dz$

$(c) \int_{0}^{1+2i} z \cos z\, dz$ $\quad$ $(d) \int_{-1+i}^{1-i} z e^{z^2}\, dz$

20. Let γ be the semicircle from $-3i$ to $3i$ in anticlockwise direction. Show that $\int_\gamma \frac{dz}{z} = \pi i$.

21. Let γ be the unit circle $|z| = 1$ traversed in the clockwise direction. Evaluate

$(a) \int_\gamma Log(z + 2)\, dz$ $\quad$ $(b) \int_\gamma \frac{dz}{3z^2 + 1}$

22. Let f be analytic within and on a positively oriented closed contour γ, and the point z_0 is not on γ. Show that

$$\int_\gamma \frac{f(z)}{(z-z_0)^2}dz = \int_\gamma \frac{f'(z)}{z-z_0}dz$$

23. Let γ be a simple closed contour described in the positive sense, and write $g(z) = \int_\gamma \frac{\xi^3+7\xi}{(\xi-z)^3}d\xi$. Show that $g(z) = 6\pi iz$ when z is inside γ and that $g(z) = 0$ when z is outside γ.

24. Let $f(z) = (e^z + e^{-z})/2$. Evaluate $\int_\gamma \frac{f(z)}{z^4}dz$, where γ any simple closed curve is enclosing zero.

25. Let γ be the contour $z = e^{i\theta}, -\pi \le \theta \le \pi$ traversed in the positive direction. Show that for any real constant a,

(a) $\displaystyle\int_\gamma \frac{e^{az}}{z}dz = 2\pi i.$

(b) Deduce that $\displaystyle\int_0^\pi e^{a\cos\theta}\cos(\text{a}\sin\theta)\,d\theta = \pi.$

26. Let γ denote the boundary of the rectangle whose vertices are $-2-2i, 2-2i, 2+i$ and $-2+i$ in the positive direction. Evaluate each of the following integrals:

(a) $\displaystyle\int_\gamma \frac{e^{-z}}{z-\frac{\pi i}{4}}dz$

(b) $\displaystyle\int_\gamma \frac{\cos z}{z^4}dz$

$$(c) \int_{\gamma} \frac{z}{(2z+1)^2} dz$$

$$(\mathrm{d}) \int_{\gamma} \frac{e^{-z}}{z^2+2} dz$$

$$(e) \int_{\gamma} \frac{dz}{z(z+1)}$$

$$(f) \int_{\gamma} \left[e^{z} \sin z + \frac{1}{(z^2+3)^2}\right] dz$$