# Lecture 7 Linear Transformations

**Def** Let $A$ and $B$ be arbitrary nonempty sets. Suppose to each element in $A$ there is assigned a unique element of $B$; the collection $f$ of such assignments is called a <u>mapping</u> from $A$ into $B$, and is denoted by

$$f: A \longrightarrow B.$$

**Remarks** * If $f: A \longrightarrow B$, $A' \subset A$, and $B' \subset B$. Then

$$f(A') = \{f(a) : a \in A'\}$$

$$f^{-1}(B') = \{a \in A : f(a) \in B'\}$$

* If $f: A \longrightarrow B$ and $g: B \longrightarrow C$ are two mappings

$$g \circ f : A \longrightarrow C, \quad g \circ f(a) = g(f(a)).$$

* Let $f: A \to B$, $g: B \to C$, $h: C \to D$. then
$h \circ (g \circ f) = (h \circ g) \circ f$

* A mapping $f: A \to B$ is said to be $\boxed{\text{one-to-one}}$ if
$f(a) = f(a') \Rightarrow a = a'$

* A mapping $f: A \to B$ is said to be $\boxed{\text{onto}}$ if every $b \in B$ is the image of at least one $a \in A$.

* A mapping $f: A \to B$ is $\boxed{\text{bijective}}$ if it one-to-one and onto.

* Let $A$ be any nonempty set. The mapping $f: A \to A$ defined by $f(a) = a$ is called the $\boxed{\text{identity mapping}}$. It is denoted by $I_A$. Thus $I_A(a) = a$.

* Let $f: A \to B$. We call $g: B \to A$ the $\boxed{\text{inverse}}$ of $f$, written $f^{-1}$, if

$$f \circ g = I_B \quad \text{and} \quad g \circ f = I_A.$$

**Def.** Let $V$ and $U$ be vector spaces over the same field $K$. A mapping $F: V \to U$ is called a $\boxed{\text{linear transformation}}$ if it satisfies the following two conditions:

① For any vectors $v, w \in V$, $F(v+w) = F(v) + F(w)$

② For any scalar $k$ and vector $v \in V$, $F(kv) = kF(v)$.

Remark: ① in ① the $+$ in $v+w$ refers to the addition in $V$, whereas the $+$ in $F(v)+F(w)$ refers to the addition operation in $U$.

(ii) In (2) the scalar product $kv$ is in $V$, while the scalar product $kF(v)$ is in $U$.

**Remark:** A Linear mapping $F: V \to U$ is completely characterized by the condition:

$$\boxed{F(av + bw) = aF(v) + bF(w),}$$

for any $a, b \in K$ and $v, w \in V$.

**Remark:** for any scalars $a_i \in K$ and any vectors $v_i \in V$, we obtain

$$F(a_1v_1 + a_2v_2 + \cdots + a_mv_m)$$
$$= a_1F(v_1) + a_2F(v_2) + \cdots + a_mF(v_m).$$

**Ex:** Let $L: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$L((u_1, u_2, u_3)) = (u_1+1, 2u_2, u_3)$$

Is $L$ a linear transformation?

Let

$u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$.

$$L(u+v) = L((u_1, u_2, u_3) + (v_1, v_2, v_3))$$
$$= L((u_1+v_1, u_2+v_2, u_3+v_3))$$
$$= ((u_1+v_1)+1, 2(u_2+v_2), u_3+v_3)$$

$$L(u) + L(v) = (u_1+1, 2u_2, u_3) + (v_1+1, 2v_2, v_3)$$
$$= (u_1+v_1+2, 2(u_2+v_2), u_3+v_3).$$

That is $L(u+v) \neq L(u) + L(v)$.

Thus $L$ is not a linear tranformation.

**Ex** Let $L : P_1 \longrightarrow P_2$ be defined

by $L[p(t)] = t\,p(t)$

Show that $L$ is a linear transformation.

**Proof** Let $p(t)$ and $q(t)$ be vectors in $P_1$ and let $c$ be a scalar,

Then $L[p(t) + q(t)] = t[p(t) + q(t)]$
$$= t\,p(t) + t\,q(t)$$
$$= L(p(t)) + L(q(t)).$$

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$$L(cp(t)) = t\,(cp(t))$$
$$= c\,(tp(t)) = c\,L(p(t)).$$

$\Rightarrow$ L is a linear transformation

**Theorem** Let $L: V \to W$ be a linear transformation. Then

(a) $L(0_V) = 0_W$.

(b) $L(u-v) = L(u) - L(v)$ for $u, v \in V$.

**Proof.** we have

$$0_V = 0_V + 0_V$$
$$\Rightarrow L(0_V) = L(0_V + 0_V)$$
$$\Rightarrow L(0_V) = L(0_V) + L(0_V)$$

Adding $-L(0_W)$ to both sides, we have

$$L(0_W) = 0_W.$$

(b) $L(u-v) = L(u+(-v)) = L(u) + L(-v)$
$= L(u) - L(v)$