Lecture 6

Linear Algebra 204R

Finite-Dimensional vector space

Theorem (1)

If $\{u_1, u_2, \dots, u_m\}$ and $\{w_1, w_2, \dots, w_n\}$ are both bases of a vector space V over a field F, then $m=n$.

Proof

Suppose that $m \neq n$. Let us say that $m<n$. Consider the set $\{w_1, u_1, u_2, \dots, u_m\}$. Since the u's span V, we know that w_1 is a linear combination of u's, say,

$w_1 = a_1 u_1 + a_2 u_2 + \dots + a_m u_m$, where a's belong to F. Clearly, not all the a's are 0. For convenience, say $a_1 \neq 0$.

Then $\{w_1, u_2, \ldots, u_m\}$ spans V. Next, consider the set $\{w_1, w_2, u_2, \ldots, u_m\}$

This time w_2 is a linear combination of $w_1, u_2, \ldots, u_m$, say

$$w_2 = b_1 w_1 + b_2 u_2 + \cdots + b_m u_m,$$

where the b's belong to F. Then at least one of $b_2, \ldots, b_m$ is nonzero, for otherwise the w's are not linearly independent. Let us say $b_2 \neq 0$. Then $w_1, w_2, u_3, \ldots, u_m$ span V. Continuing in this fashion, we see that $\{w_1, w_2, \ldots, w_m\}$ spans V. But then w_{m+1} is a linear combination of $w_1, w_2, \ldots, w_m$, and therefore the set $\{w_1, \ldots, w_n\}$ is not linearly independent. This contradiction finishes the proof.

Remark

Theorem (1) Shows that any two finite bases for a vector space have the same size.

Theorem (2)

Let V be a vector space of finite-dimension and let $S = \{u_1, u_2, \ldots, u_r\}$ be a set of linearly independent vectors in V. Then S is a part of a basis of V, that is S may be extended to a basis of V.

Proposition (3)

If V is finite dimensional, then every linearly independent list of vectors in V with length $\dim V$ is a basis of V.

Remarks (4)

Let V be a vector space of finite dimension n. Then

(1) any $n+1$ or more vectors must be linearly dependent.

(2) Let W be a subspace of V. Then $\dim W \leq n$. In particular, if $\dim W = n$, then $W = V$.

Theorem (5)

If U and W are finite-dimensional subspaces of a vector space V, then $U+W$ is finite-dimensional and

$$\dim U + \dim W = \dim(U+W) + \dim(U \cap W).$$

proof.

Observe that $U \cap W$ is a subspace of both U and W. Suppose $\dim U = m$, $\dim W = n$, and $\dim(U \cap W) = r$. Suppose $\{v_1, v_2, --, v_r\}$ is a basis of $U \cap W$. We can extend $\{v_i\}_{1 \le i \le r}$ to a basis of U and to a basis of W;

Say $\{v_1, \dots, v_r, u_1, \dots, u_{m-r}\}$ and

$\{v_1, \dots, v_r, w_1, \dots, w_{n-r}\}$

are bases of U and W respectively.

Let $B = \{v_1, \dots, v_r, u_1, \dots, u_{m-r}, w_1, \dots, w_{n-r}\}$.

Note that B has exactly $m+n-r$ elements. Thus the theorem is proved if we can show that B is a basis of $U+W$. Since $\{v_i, u_j\}$ spans U and $\{v_i, w_k\}$ spans W, the union $B = \{v_i, u_j, w_k\}$ spans $U+W$.

Thus it suffices to show that B is independent.

Suppose $a_1v_1 + a_2v_2 + \dots + a_rv_r + b_1u_1 + \dots + b_{m-r}u_{m-r} + c_1w_1 + \dots + c_{n-r}w_{n-r} = 0$, where a_i, b_j, c_k are scalars.

Thus, we have

$$\sum_{i=1}^{r} a_i v_i + \sum_{j=1}^{m-r} b_j u_j + \sum_{k=1}^{n-r} c_k w_k = 0 \quad (1)$$

$$\Rightarrow -\sum_{k=1}^{n-r} c_k w_k = \sum_{i=1}^{r} a_i v_i + \sum_{j=1}^{n-r} b_j u_j .$$

Hence $\sum_{k=1}^{n-r} c_k w_k \in W$ and

$\sum_{k=1}^{n-r} c_k w_k \in U$. Thus

$\sum_{k=1}^{n-r} c_k w_k \in U \cap W$. Therefore

$$\sum_{k=1}^{n-r} c_k w_k = \sum_{i=1}^{r} \alpha_i v_i .$$

Because $\{v_1, \dots, v_r, w_1, \dots, w_{n-r}\}$ is independent, $c_k = 0$; for all k. Hence from (1), we have

$$\sum_{i=1}^{r} a_i v_i + \sum_{j=1}^{m-r} b_j u_j = 0.$$

Since $\{v_1, \ldots, v_r, u_1, \ldots, u_{m-r}\}$ is linearly independent. Then

$a_i, b_j = 0$ for all i, j.

Thus $\{v_i, u_j, w_k\}_{\substack{1 \le i \le r \\ 1 \le j \le m-r \\ 1 \le k \le n-r}}$ is linearly independent.

<u>Example</u> (6) Let $V = M_{2\times 2}$, the vector space of 2×2 matrices,

Let $U = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \right\}$, $W = \left\{ \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} \right\}$

$U + W = \left\{ \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \right\}$, $U \cap W = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} \right\}$

$$\begin{aligned} \dim(U+W) &= \dim U + \dim W - \dim(U \cap W) \\ &= 2 + 2 - 1 = 3. \end{aligned}$$