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Lecture 8 — Linear Transformation

Def A linear transformation $L: V \longrightarrow W$ is called one-to-one (onto) if it is one-to-one (onto).

Ex Let $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $L(u_1, u_2) = (u_1 + u_2, u_1 - u_2)$

To determine whether L is one-to-one or not. Let

$v_1 = (u_1, u_2)$, $v_2 = (v_1, v_2)$ and

$$L(v_1) = L(v_2) \Rightarrow u_1 + u_2 = v_1 + v_2,$$
$$\Rightarrow u_1 - u_2 = v_1 - v_2$$

Adding equations

$$2u_1 = 2v_1 \Rightarrow u_1 = v_1,$$
$$u_2 = v_2.$$

Def. Let $L: V \longrightarrow W$ be a linear transformation. The kernel of L, $\ker L$, is defined by

$$\ker L = \{v \in V : L(v) = 0_W\}.$$

Remark:

Ker L is never an empty set because If $L: V \to W$ is a linear transformation, then $0_V \in \text{Ker} L$.

Ex Let $L: \mathbb{R}^3 \to \mathbb{R}^2$,

$L(u_1, u_2, u_3) = (u_1, u_2)$

$L(0,0,2) = (0,0)$

$\Longrightarrow (0,0,2) \in \text{Ker} L.$

Theorem

Let $L: V \to W$ be a linear transformation. Then

(a) Ker L is a subspace of V

(b) L is one-to-one $\Longleftrightarrow$ $\text{Ker} L = \{0_V\}$.

proof: (a) If $v, w \in \text{Ker} L$, then

$L(v) = 0_W$ and $L(w) = 0_W$.

Hence

$$L(v+w) = L(v) + L(w) = 0_W + 0_W = 0_W$$

Thus $v+w \in \operatorname{Ker} L$. Also,

$$L(cw) = cL(v) = c\,0_W = 0_W$$

Therefore, $cw \in \operatorname{Ker} L$, for any scalar c.

(b) $\Rightarrow$ Let L is one-to-one, we show that $\operatorname{Ker} L = \{0_V\}$.

Let $v \in \operatorname{Ker} L \Rightarrow L(v) = 0_W = L(0_V)$.

Therefore $v = 0_V$. Thus

$\operatorname{Ker} L = \{0_V\}$.

$\Leftarrow$ Let $\operatorname{Ker} L = \{0_V\}$, we show that L is one-to-one.

Let $L(v_1) = L(v_2) \Rightarrow L(v_1) - L(v_2) = 0_W$

$\Rightarrow L(v_1 - v_2) = 0_W$

$\Rightarrow v_1 - v_2 \in \operatorname{Ker} L$

$\Rightarrow v_1 - v_2 = 0_V$

$\Rightarrow v_1 = v_2$.

Remark

We can state (b) of the previous theorem as follows:

L is one-to-one $\iff$ dim Ker L = 0.

Def. If $L: V \to W$ is a linear transformation. Then the Range of L or image of V is defined by:

$$\operatorname{Im} L = \{ u \in W \mid \text{there exists } v \in V \text{ s.t } L(v) = u \}$$

Theorem

If $L: V \to W$ is a Linear transformation, then the range L is a subspace of W.

Proof.

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Let $w_1, w_2 \in \operatorname{Im} L$. Then there exist $v_1, v_2 \in V$ s.t $L(v_1) = w_1$ and $L(v_2) = w_2$. Thus

$$w_1 + w_2 = L(v_1) + L(v_2) = L(v_1 + v_2).$$

Hence

$$w_1 + w_2 \in \operatorname{Im} L.$$

Also, if $w \in \operatorname{Im} L$, then there exists v s.t $L(v) = w$. Thus

$cw = cL(v) = L(cv)$, where c is a scalar. Thus $cw \in \operatorname{Im} L$.

Theorem: Suppose $v_1, v_2, \dots, v_m$ span a vector space V, and suppose $F: V \longrightarrow U$ is linear. Then $F(v_1), F(v_2), \dots, F(v_m)$ span $\operatorname{Im} F$.

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Def. Let $F: V \to U$ be a linear transformation. The rank of F is defined to be the dimension of its image, and the nullity of F is defined to be the dimension of its kernel:

rank(F) = dim(Im F) and

nullity(F) = dim(Ker F).

Theorem If $L: V \to W$ is a linear transformation of an n-dimensional vector space V, in to a vector space W, then dim ker L + dim range L = dim V.

proof

Let $k = \dim \ker L$.

If $k = n$, then $\ker L = V \Rightarrow L(v) = 0_W$ for every $v \in V$. Hence $\text{Range } L = \{0_W\}$

$\Rightarrow \dim \text{Range } L = 0$.

Next, suppose that $1 \leq k < n$. We shall prove that $\dim \text{Range } L = n - k$. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for $\ker L$.

We can extend this basis to a basis $S = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V.

We prove that the set

$T = \{L(v_{k+1}), L(v_{k+2}), \ldots, L(v_n)\}$ is a basis for Range L.

First we show that T spans Range L.

Let w be any vector in Range L.

Then $w = L(v)$ for some $v \in V$. Since S is a basis for V, then

$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$

Then

$$w = L(v) = L(a_1 v_1 + a_2 v_2 + \dots + a_n v_n)$$

$$= a_{k+1} L(v_{k+1}) + \dots + a_n L(v_n)$$

$\Rightarrow$ T spans Range L.

Now we show that T is linearly independent. Suppose that

$a_{k+1} L(v_{k+1}) + a_{k+2} L(v_{k+2}) + \dots + a_n L(v_n) = 0_W$

$\Rightarrow L(a_{k+1} v_{k+1} + a_{k+2} v_{k+2} + \dots + a_n v_n) = 0_W$

$\Rightarrow a_{k+1} v_{k+1} + a_{k+2} v_{k+2} + \dots + a_n v_n \in \text{Ker } L$

$\Rightarrow a_{k+1} v_{k+1} + a_{k+2} v_{k+2} + \dots + a_n v_n =$

$b_1 v_1 + b_2 v_2 + \dots + b_k v_k$

$\Rightarrow a_{k+1} v_{k+1} + a_{k+2} v_{k+2} + \dots + a_n v_n - b_1 v_1 - b_2 v_2 - \dots - b_k v_k = 0.$

Since S is linearly independent, we find that

$$b_1 = b_2 = \cdots = b_k = a_{k+1} = \cdots = a_n = 0.$$

Thus T is linearly independent

Ex Let $F : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear mapping defined by

$$F(x, y, z, t) = (x - y + z + t,\ 2x - 2y + 3z + 4t,\ 3x - 3y + 4z + 5t)$$

(a) Find a basis and the dimension of the image of F.

(b) Find a basis and the dimension of the kernel of the map F.

Sol First find the image of the usual basis vectors in $\mathbb{R}^4$.

$F(1,0,0,0) = (1,2,3)$, $F(0,1,0,0) = (-1,-2,-3)$, $F(0,0,1,0) = (1,3,4)$, $F(0,0,0,1) = (1,4,5)$

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The image vectors span Im F.

Let $M = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 1 & 3 & 4 \\ 1 & 4 & 5 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ (1, 2, 3) and (0, 1, 1) form a basis of Im F $\Rightarrow$ dim (Im F) = 2

Thus dim Ker F = 4 − 2 = 2.

(b) Basis for kernel.

$F(x, y, z, t) = (x - y + z + t,\ 2x - 2y + 3z + 4t,\ 3x - 3y + 4z + 5t)$

$= (0, 0, 0)$

$$\begin{aligned} x - y + z + t &= 0 \\ 2x - 2y + 3z + 4t &= 0 \\ 3x - 3y + 4z + 5t &= 0 \end{aligned} \Rightarrow \begin{aligned} x - y + z + t &= 0 \\ z + 2t &= 0 \\ z + 2t &= 0 \end{aligned}$$

$$\Rightarrow \begin{aligned} x - y + z + t &= 0 \\ z + 2t &= 0 \end{aligned}$$

Theorem The dimension of the solution space W of a homogenous system $AX=0$ is $n-r$ where n is the number of unknowns and r is the rank of the coefficient matrix A

The free variables are y and t.

Hence $\dim \ker F = 2$.

(i) Set $y=1$, $t=0$ $\xrightarrow{sol}$ $(-1,1,0,0)$

(ii) Set $y=0$, $t=1$ $\xrightarrow{sol}$ $(1,0,-2,1)$.

Thus $(-1,1,0,0)$ and $(1,0,-2,1)$ form a basis for $\ker F$.

Operations with linear mappings

Let $F: V \to U$ and $G: V \to U$ be linear mappings over K.

The sum $F+G$ is defined by:

$$(F+G)(v) = F(v) + G(v).$$

The scalar product kF, where $k \in K$ is defined by:

$$(kF)(v) = k\,F(v).$$