2

SPECIAL FUNCTIONS–II

2.1 LEGENDRE'S EQUATION AND LEGENDRE'S FUNCTION

The differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)y = 0 \qquad (2.1)$$

is called the Legendre's equation and the solution to this differential equation is called the Legendre's function.

Equation (2.1) can also be written as

$$\frac{d}{dx}\left[(1 - x^2)\frac{dy}{dx}\right] + n(n + 1)y = 0$$

We will find the solution in series.

Assume that $$y = \sum_{r=0}^{\infty} a_r x^{k-r} \qquad (2.2)$$

i.e. $$y = a_0x^k + a_1x^{k-1} + a_2x^{k-2} + \cdots$$

Differentiating Eq. (2.2) with respect to x

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r(k - r)x^{k-r-1}$$

Differentiating again

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r(k - r)(k - r - 1)x^{k-r-2} \qquad (2.3)$$

Substituting these in Eq. (2.1)

$$(1-x^2)\left[\sum_{r=0}^{\infty} a_r(k-r)(k-r-1)x^{k-r-2}\right]-2x\left[\sum_{r=0}^{\infty} a_r(k-r)x^{k-r-1}\right]$$
$$+\,n(n+1)\sum_{r=0}^{\infty} a_r x^{k-r}=0$$

$$\sum_{r=0}^{\infty}[a_r(k-r)(k-r-1)x^{k-r-2}-a_r(k-r)(k-r-1)x^{k-r}$$
$$-2a_r(k-r)x^{k-r}+n(n+1)a_r]x^{k-r}=($$

$$\sum_{r=0}^{\infty}\{a_r(k-r)(k-r-1)x^{k-r-2}-a_r[(k-r)(k-r+1)-n(n+1)]\}x^{k-r}=($$

Comparing the coefficient of x^k

$$a_0[n(n+1)-k(k+1)]=0$$
$$n^2+n-k^2-k=0$$
$$(n-k)(n-k+1)=0$$

$\therefore$ $$k=n \text{ and } k=-(n+1) \qquad (2.4$$

Comparing the coefficient of x^{k-1}

$$a_1[k(k-1)-n(n+1)]=0$$
$$a_1(k\neq n)(k-n-1)=0$$

Since $k+n\neq 0,\ k-n-1\neq 0$ [From Eq. (2.4)

$$a_1=0$$

Equating the coefficient of x^{k-r-2} to zero to get the recurrence relatio between the coefficients. Our aim is to find a_0, a_1, a_2, a_3, ...

$$a_r(k-r)(k-r-1)-a_{r+2}[(k-r-2)(k-r-1)-n(n+1)]=0$$

$$a_{r+2}=\frac{a_r(k-r)(k-r-1)}{(k-r-1)(k-r-2)-n(n+1)}$$

$$(k-r-1)(k-r-2)-n^2-n=(k-r)^2-3(k-r)+2-n^2-n$$
$$=(k-r)^2-3(k-r)-(n+2)(n-1)$$
$$=(k-r)$$

When $k=n$

$$a_{r+2}=\frac{a_r(n-r)(n-r-1)}{(n-r-1)(n-r-2)-n^2-n}$$

$$=\frac{a_r(n-r)(n-r-1)}{-2nr-4n+r^2+2+3r}=\frac{(n-r)(n-r-1)a_r}{-(2n-r-1)(r+2)}$$

Put $r = 0$

$$\therefore \qquad a_2 = -a_0 \frac{n(n-1)}{2\cdot(2n-1)}$$

By substituting these values in $a_4, a_6, a_8 \ldots$

$$a_4 = \frac{-(n-2)(n-3)}{4(2n-3)} a_2 = \frac{-n(n-1)(n-2)(n-3)a_0}{2\cdot 4(2n-1)(2n-3)}$$

$$a_6 = \frac{-n(n-1)(n-2)(n-3)(n-4)(n-5)a_0}{2\cdot 4\cdot 6\,(2n-1)(2n-3)(2n-5)}$$

$$\vdots$$

$$a_{2r} = (-1)^r \frac{n(n-1)\ldots(n-2r+1)a_0}{2\cdot 4\cdot 6\ldots 2r(2n-1)\ldots(2n-2r+1)}$$

Substituting these values in y

$$y = a_0\left[x^n - \frac{n(n-1)}{2\cdot(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2\cdot 4\cdot 6(2n-1)(2n-3)(2n-5)}x^{n-6} + \cdots\right]$$

is a solution of Eq. (2.1) in descending powers of x.

When $k = -(n+1)$

$$a_{r+2} = \frac{(n+r+1)(n+r+2)a_r}{(r+2)(2n+r+3)}$$

Substituting $r = 0, 2, 4, \ldots$ we get

$$a_2 = \frac{(n+1)(n+2)a_0}{2\cdot(2n+3)},\ a_4 = \frac{(n+3)(n+4)a_2}{4(2n+5)},\ a_6 = \frac{(n+5)(n+6)a_4}{6(2n+7)}$$

$$\vdots$$

$$a_{2r} = \frac{(n+1)(n+2)(n+3)(n+4)\ldots(n+2r)a_0}{2\cdot 4\ldots 2r(2n+3)(2n+5)(2n+7)\ldots(n+2r+1)}$$

Substituting these values in y

$$y = a_0\left[x^{-n-1} + \frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4(2n+3)(2n+5)}x^{-n-5} + \cdots\right]$$

is a solution in descending powers of x.

2.1.1 Legendre's Function

The two independent solutions of Eq. (2.1) for $k = -n$ and $k = -(n+1)$ are called Legendre's functions.

Legendre's function of first kind

We have found the solution of

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)y = 0$$

$$y = a_0\left[x^n - \frac{n(n-1)x^{n-2}}{2(2n-1)} + \frac{n(n-1)(n-2)(n-3)x^{n-4}}{2\cdot 4\cdot(2n-1)(2n-3)} - \ldots\right]$$

where a_0 is a constant and n is a positive integer

If $a_0 = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}$, y is called a Legendre's function of first kind and written as $P_n(x)$

$$y = P_n(x) = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}$$

$$\left[x^n - \frac{n(n-1)x^{n-2}}{2(2n-1)} + \frac{n(n-1)(n-2)(n-3)x^{n-4}}{2\cdot 4\cdot(2n-1)(2n-3)} \cdots\right]$$

For different values of n, we get the Legendre's polynomials.

Note: When n is even, we get $\frac{n}{2} + 1$ terms

When n is odd, we get $\frac{n+1}{2}$ terms

Legendre's function of second kind

The solution of Eq. (2.1) when $k = -(n + 1)$ and

$$a_0 = \frac{n!}{1\cdot 3\cdot 5\ldots(2n+1)}$$

is called the Legendre's function of second kind and written as $Q_n(x)$

$$Q_n(x) = \frac{n!}{1\cdot 3\cdot 5\ldots(2n+1)}$$

$$\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4(2n+3)(2n+5)}x^{-n-5} + \cdots\right]$$

2.1.2 Rodrigue's Formula

$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$ is called the Rodrigue's formula.

Now we will prove that this formula satisfies the differential equation; i.e. Eq. (2.1). **(JNTU 1989)**

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0$$

Proof: Let $V=(x^2-1)^n$

$$\frac{dV}{dx}=n(x^2-1)^{n-1}\,2x$$

Multiplying both sides by (x^2-1)

$$(x^2-1)\frac{dV}{dx}=2nx(x^2-1)^n=2nxV$$

Differentiating again with respect to x

$$(x^2-1)\frac{d^2V}{dx^2}+2x\frac{dV}{dx}=2n\left(V+x\frac{dV}{dx}\right)$$

$$(x^2-1)\frac{d^2V}{dx^2}+2x(1-n)\frac{dV}{dx}-2nV=0$$

Differentiating n times using Leibnitz's formula

$$(x^2-1)\,V_{n+2}+n\cdot 2x\,V_{n+1}+\frac{n(n-1)}{2}\cdot 2V_n$$
$$+\,2x(1-n)V_{n+1}+2n(1-n)V_n-2nV_n=0$$

[*Note:* Leibnitz's formula

$$D^n(uv)=uD^nv+nC_1Du\,D^{n-1}v+nC_2D^2uD^{n-2}v+\cdots nC_nD^nu\cdot v]$$

where V_n, V_{n+1}, V_{n+2} are nth, $(n+1)$th, $(n+2)$th differentials of v

$$(x^2-1)V_{n+2}+2xV_{n+1}-n(n+1)V_n=0$$

$$(1-x^2)V_{n+2}-2xV_{n+1}+n(n+1)V_n=0$$

$\therefore$ V_n is a solution of Eq. (2.1)

Suppose $P_n(x)=CV_n$ where C is a constant.

To find the value of C

If $x=1$

$P_n(1)=1$

$$1=(V_n)_{x=1}=C\frac{d^n}{dx^2}(x^2-1)^n$$

$$\frac{d^n}{dx^n}(x^2-1)^n=(x-1)^n\frac{d^n}{dx^n}(x+1)^n+n\cdot n(x-1)^{n-1}$$

$$\frac{d^{n-1}}{dx^{n-1}}(x+1)^n + nC_2 n(n-1)(x-1)^{n-2}\frac{d^{n-2}}{dx^{n-2}}(x+1)^n + \cdots$$

$$\cdots + (x+1)^n \frac{d^n(x-1)^n}{dx^n}$$

When $x = 1$ $\qquad \frac{d^nV}{dx^n} = 2^n\, n! \qquad \left[\because \frac{d^n(x-1)^n}{dx^n} = n!\right]$

$\therefore$ $\qquad 1 = 2^n\, n! C$

$\therefore$ $\qquad C = \frac{1}{2^n n!}$

$\therefore$ $\qquad P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

To find $P_n(x)$, $n = 0, 1, 2,$

$$P_0(x) = \frac{1}{2^0 \cdot 0!}\cdot 1 = 1$$

$$\therefore\ P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = \frac{2x}{2} = x$$

$$P_2(x) = \frac{1}{2^2 \cdot 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d^2}{dx^2}(x^4-2x^2+1)$$

$$= \frac{1}{8}(12x^2-4) = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1}{2^3 \cdot 3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}\frac{d^3}{dx^3}(x^6-3x^4+3x^2-1)$$

$$= \frac{1}{48}(120x^3-72x)$$

$$= \frac{x(5x^2-3)}{2}$$

Similarly, we can find $P_4(x)$, $P_5(x)$ ...

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$

$$P_5(x) = \frac{1}{8}(63x^5-70x^3-15x)$$

To express polynomials in terms of $P_n(x)$

$$1 = P_0(x),\ x = P_1(x)$$

$$P_2(x) = \frac{3x^2-1}{2} = \frac{3x^2-P_0(x)}{2}$$

$$\therefore \quad x^2 = \frac{2P_2(x) + P_0(x)}{3}$$

$$P_3(x) = \frac{5x^3 - 3x}{2} = \frac{5x^3 - 3P_1(x)}{2}$$

(JNTU 2006 April, 2008 Nov.)

$$\therefore \quad x^3 = \frac{2P_3(x) + 3P_1(x)}{5}$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$ **(JNTU 2006 Nov.)**

$$= \frac{1}{8}\left[35x^4 - 30\left(\frac{2P_2(x) + P_0(x)}{3}\right) + 3P_0(x)\right]$$

$$= \frac{1}{8}[35x^4 - 20P_2(x) - 7P_0(x)]$$

$$\therefore \quad x^4 = \frac{8P_4(x) + 20P_2(x) + 7P_0(x)}{35}$$

WORKED-OUT PROBLEMS

1. Express $x^3 - 2x^2 + 1$ in terms of Legendre polynomials.

Solution: $P_0(x) = 1,\ x^2 = \dfrac{2P_2(x) + P_0(x)}{3}$ and $x^3 = \dfrac{2P_3(x) + 3P_1(x)}{5}$

$$\therefore \quad x^3 - 2x^2 + 1 = \frac{2P_3(x) + 3P_1(x)}{3} - 2\frac{(2P_2(x) + P_0(x))}{3} + P_0(x)$$

$$= \frac{2P_3(x) - 4P_2(x) + 3P_1(x) + P_0(x)}{3}.$$

2. Prove that $\int_{-1}^{1} P_n(x)dx = 0$ except when $n = \theta$.

Solution: Rodrigue's formula is $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2 - 1)^n$

$$\int_{-1}^{1} P_n(x)dx = \int_{-1}^{1} \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n\, dx$$

$$= \frac{1}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}(x^2 - 1)^n \Big|_{-1}^{1} = 0 \text{ (when } n \neq 0)$$

If $n = 0$ $P_0(x) = 1$

$$\int_{-1}^{1} P_0(x)dx = \int_{-1}^{1} dx = 2.$$

3. Prove that $P_n(0) = 0$ for n is odd.

Solution:
$$P_n(x) = \frac{1\cdot 3\cdot 5 \ldots (2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot (2n-1)(2n-3)}x^{n-4} + \ldots\right]$$

If $n = 2m + 1$

$$P_{2m+1}(x) = \frac{1\cdot 3\cdot 5 \ldots (2m+1)}{(2m+1)!}\left[x^{2m+1} - \frac{(2m+1)2m}{2\cdot 2m+1}x^{2m-1} \ldots\right]$$

Substituting $x = 0$

$$P_{2m+1}(0) = 0$$

Therefore, if n is odd

$$P_n(x) = 0.$$

2.1.3 Generating Function

The function which generates $P_n(x)$, $n = 1, 2, 3, \ldots$ is called the generating function for $P_n(x)$. The generating function for $P_n(x)$ is $(1 - 2xz + z^2)^{-1/2}$. We will prove that $P_n(x)$ is the coefficient of z^n in the expansion of $(1 - 2xz + z^2)^{-1/2}$.

(JNTU 2006 April, 2006 Nov., 2008 April/May)

Proof:
$$(1 - 2xz + z^2)^{-1/2} = \sum z^n P_n(x)$$

Expanding binomically

$$[1 - z(2x - z)]^{-1/2} = 1 + \frac{1}{2}z(2x - z) + \frac{1}{2}\left(\frac{3}{2}\right)\frac{z^2}{2!}(2x - z)^2$$

$$+ \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\frac{z^3}{3!}(2x - z)^3 + \cdots \frac{1}{2}\frac{3}{2} \cdots \frac{(2n-1)}{2\cdot n!}z^n(2x - z)^n + \cdots$$

$$= 1 + xz + \left(\frac{1.3}{2^2}x^2 - \frac{1}{2}\right)z^2 + \left[\frac{(1\cdot 3;5x^3 - 3x)}{2}\right]z^3$$

$$+ \cdots \frac{1\cdot 3 \ldots 2n-1}{n!}\left(x^n - \frac{n(n-1)}{2\cdot 2n-1}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)} + \ldots\right)$$

$$P_0(x) + P_1(x)z + P_2(x)z^2 + \cdots + P_n(x)z^n + \cdots$$

$\therefore$
$$(1 - 2xz + z^2)^{-1/2} = \sum_0^\infty z^n P_n(x)$$

$\therefore$ $(1 - 2xz + z^2)^{-1/2}$ is the generating function for $P_n(x)$.

2.1.4 Recurrence Relations

(1) $(2n + 1)xP_n = (n + 1)P_{n+1}(x) + (n)P_{n-1}(x)$

(JNTU 2001, 2005, 2007 April, 2008 Aug./Sep.)

$$(1 - 2xz + z^2)^{-1/2} = \sum_0^{\infty} z^n P_n(x) \tag{2.5}$$

Differentiating both sides with respect to z

$$\frac{-1}{2}(1 - 2xz + z^2)^{-3/2}(-2x + 2z) = \sum nz^{n-1}P_n(x)$$

$$\frac{x-z}{(1-2xz+z^2)^{3/2}} = \sum nz^{n-1}P_n(x) \tag{2.6}$$

Multiplying both sides by $(1 - 2xz + z^2)$

$$\frac{x-z}{(1-2xz+z^2)^{1/2}} = \sum nz^{n-1}(1 - 2xz + z^2)P_n(x) \tag{2.7}$$

$$= \sum [nz^{n-1}P_n(x) - 2xnP_n(x)z^n + nP_n(x)z^{n+1}]$$

$$(x-z)\sum P_n z^n = \sum nP_n(x)z^{n-1} - \sum 2xnP_n(x)z^n + \sum nP_n(x)z^{n+1}$$

Since $(1 - 2xz + z^2)^{-1/2} = \sum P_n(x)z^n$

Equating the coefficient of z^n on both sides

$$xP_n - P_{n-1} = (n+1)P_{n+1} - 2xP_n(x)n + (n-1)P_{n-1}(x)$$

$$(2n+1)xP_n = (n+1)P_{n+1}(x) + nP_{n-1}(x) \tag{2.8}$$

(2) $xP_n' - P'_{n-1} = nP_n$

The generating function for $P_n(x)$ is $(1 - 2xz + z^2)^{-1/2}$

$$(1 - 2xz + z^2)^{-1/2} = \sum_0^{\infty} P_n(x)z^n$$

Differentiating Eq. (2.5) with respect to x on both sides

$$\frac{-1}{2}(1 - 2xz + z^2)^{-3/2}(-2z) = \sum P_n'(x)z^n \tag{2.9}$$

$$\frac{z}{(1-2xz+z^2)^{1/2}} = \sum (1 - 2xz + z^2)P_n' z^n \tag{2.10}$$

Dividing Eq. (2.7) by Eq. (2.10)

$$\frac{(x-z)}{z} = \frac{\sum nz^{n-1}P_n(x)}{\sum P_n' z^n}$$

$$\sum (P_n' xz^n - P_n' z^{n+1}) = \sum nz^n P_n(x) \tag{2.11}$$

Comparing the coefficient of t^n on both sides

$$xP_n' - P'_{n-1} = nP_n$$

(3) $(2n + 1)P_n = P'_{n+1} - P'_{n-1}$ **(JNTU 2008 April/May, Nov.)**

Recurrence formula (1) is

$$(2n + 1)xP_n = (n + 1)P_{n+1}(x) + nP_{n-1}(x)$$

Differentiating with respect to x

$$(2n + 1)P_n + (2n + 1)xP_n' = (n + 1)P'_{n+1} + nP'_{n-1} \quad (2.12)$$

From recurrence relation (2)

$$xP_n' = nP_n + P'_{n-1} \quad (2.13)$$

Substituting the value of xP_n' from Eq. (2.13) in Eq. (2.12)

$$(2n + 1)P_n + (2n + 1)(nP_n + P'_{n-1}) = (n + 1)P'_{n+1} + nP'_{n-1}$$

$$(2n + 1)(n + 1)P_n + (n + 1)P'_{n-1} = (n + 1)P'_{n+1}$$

$$(2n + 1)P_n = P'_{n+1} - P'_{n-1} \quad (2.14)$$

(4) $(n + 1)P_n = P'_{n+1} - xP_n'$ **(JNTU 2008 April/May)**

Recurrence relation (3), is $(2n + 1)P_n = P'_{n+1} - P'_{n-1}$

Substituting P'_{n-1} from recurrence relation (2)

$$(2n + 1)P_n = P'_{n+1} - xP_n' + nP_n$$

$$(n + 1)P_n = P'_{n+1} - xP_n' \quad (2.15)$$

(5) $(1 - x^2)P_n' = n(P_{n-1} - xP_n)$

(JNTU 2001 S, 2004 Nov., 2007 Feb.)

Recurrence relation (4) is

$$(n + 1)P_n = P'_{n+1} - xP_n'$$

Recurrence relation (2) is

$$xP_n' = nP_n + P'_{n-1}$$

Substituting $(n - 1)$ for n in Eq. (2.15)

$$nP_{n-1} = P_n' - xP'_{n-1} \quad (2.16)$$

Multiplying Eq. (2.13) by x and subtracting from Eq. (2.16)

$$P_n' - xP'_{n-1} - x^2P_n' = nP_{n-1} - nxP_n - xP'_{n-1}$$

$$(1 - x^2)P_n' = n(P_{n-1} - xP_n)$$

(6) $(1 - x^2)P_n' = (n + 1)(xP_n - P_{n+1})$

(JNTU 2006 Aug., 2007 Feb., 2007 Nov.)

Recurrence formula (1)

$$(2n + 1)xP_n = (n + 1)P_{n+1} + nP_{n-1}$$

$$(n + 1)xP_n = (n + 1)P_{n+1} + nP_{n-1} - nxP_n$$

$$(n + 1)(xP_n - P_{n+1}) = nP_{n-1} - nxP_n$$

$$= n(P_{n-1} - xP_n)$$

$$= (1 - x^2)P_n' \quad \text{[from recurrence relation (5)]}$$

$$\therefore \quad (1 - x^2)P_n' = (n + 1)(xP_n - P_{n+1})$$

2.1.5 Orthogonality of $P_n(x)$

(JNTU 1992, 2001, 2007 Nov., 2007 April, 2008 April/May, 2009 May/June)

$$\int_{-1}^{1} P_m(x)\, P_n(x) dx = 0 \quad (\text{if } n \neq m)$$

$$= \frac{2}{2n+1} \quad (\text{if } n = m)$$

Proof:

$$P_n(x) = \frac{1}{2^n \cdot n!} \frac{d^n (x^2 - 1)^n}{dx^n} \tag{2.17}$$

$$P_m(x) = \frac{1}{2^m \cdot m!} \frac{d^m (x^2 - 1)^m}{dx^m} \tag{2.18}$$

Multiplying Eqs. (2.17) and (2.18) and integrating between –1 and 1 with respect to x

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \int_{-1}^{1} \frac{1}{2^{n+m} n!\, m!} \frac{d^n (x^2 - 1)}{dx^n} \frac{d^m (x^2 - 1)^m}{dx^m} dx$$

$$= \frac{1}{2^{n+m} n!\, m!} \left[\frac{d^n (x^2 - 1)^n}{dx^n} \frac{d^{m-1} (x^2 - 1)^m}{dx^{m-1}} \right]_{-1}^{1}$$

$$-n\int_{-1}^{1} \left[\frac{d^{m-1} (x^2 - 1)^m}{dx^{m-1}} \frac{d^n (x^2 - 1)^n}{dx^n} \right] dx \quad \text{(Integrating by parts)}$$

$$= 0 \quad (\text{if } n \neq m)$$

If $n = m$

$$\int_{-1}^{1} P_n^{\,2}(x) dx = \int_{-1}^{1} \frac{1}{2^{2n} (n!)^2} \frac{d^n (x^2 - 1)}{dx^n} \frac{d (x^2 - 1)^n}{dx^n} dx$$

$$= \frac{1}{2^{2n} (n!)^2} \left[\frac{d^n (x^2 - 1)^n}{dx^n} \frac{d^{n-1} (x^2 - 1)^n}{dx^{n-1}} \right]_{-1}^{1}$$

$$- \int_{-1}^{1} \frac{d^{n-1} (x^2 - 1)^n}{dx^{n-1}} \cdot \frac{d^{n+1} (x^2 - 1)^n}{dx^{n+1}}$$

$$= \frac{1}{2^{2n}(n!)^2}\left[0 - \int_{-1}^{1} \frac{d^{n-1}(x^2-1)^n}{dx^{n-1}} \frac{d^{n+1}}{dx^{n+1}}(x^2-1)^n\,dx\right]$$

Integrating again

$$= \frac{1}{2^{2n}(n!)^2}\left[\int_{-1}^{1} \frac{d^{n-2}(x^2-1)^n}{dx^{n-2}} \frac{d^{n+2}}{dx^{n+2}}(x^2-1)^n\,dx\right]$$

Integrating n times

$$\int_{-1}^{1} P_n^2(x)dx = \frac{1}{2^{2n}(n!)^2} \int_{-1}^{1} \frac{d^{2n}(x^2-1)^n}{dx^{2n}} \frac{d^{n-n}(x^2-1)^n}{dx^{n-n}}$$

$$\frac{d^{2n}}{dx^{2n}}(x^2-1)^n = (2n)!$$

$$\therefore \qquad \int_{-1}^{1} P_n^2(x)dx = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2} \int_{-1}^{1}(x^2-1)^n dx$$

$$= \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\, 2\int_{0}^{1}(x^2-1)^n dx \qquad [\because\ (x^2-1)^n \text{ is even}]$$

Put $x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$\therefore \qquad \int_{-1}^{1} P_n^2(x)dx = \frac{(-1)^n(2n)!(-1)^n}{2^{2n}(n!)^2}\, 2\int_{0}^{1}\cos^{2n+1} d\theta$$

$$= \frac{(2n)!}{2^{2n}(n!)^2}\, 2\int_{0}^{1}\cos^{2n+1}\theta\, d\theta \qquad \left[\begin{aligned} \because\ 2n+1 &= 2p-1 \\ p &= n+1 \\ 2q-1 &= 0 \\ q &= \frac{1}{2} \end{aligned}\right]$$

$$= \frac{(2n)!}{2^{2n}(n!)^2}\, \beta\left(n+1, \frac{1}{2}\right)$$

$$= \frac{(2n)!}{2^{2n}(n!)^2}\, \frac{\Gamma(n+1)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{3}{2}\right)}$$

$$= \frac{(2n)!}{2^{2n}(n!)^2}\, \frac{n!\,\Gamma\left(\frac{1}{2}\right)}{\frac{(2n+1)}{2}\,\frac{(2n-1)}{2}\cdots\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{(2n)!}{2^{2n}n!}\, \frac{2^{n+1}\, 2\cdot 4\cdot 6 \ldots 2n}{1\cdot 2\cdot 3 \ldots 2n\cdot 2n+1}$$

(Multiplying and dividing by $2 \cdot 4 \cdot 6 \ldots 2n$)

$$= \frac{(2n)!\,2^n \cdot 2\,n!}{2^n \cdot n!\,2n!\,2n+1}$$

$$= \frac{2}{2n+1}$$

$$\therefore \quad \int_{-1}^{1} P_m(x)P_n(x)\,dx = \frac{2}{2n+1} \quad (\text{if } m = n)$$

WORKED-OUT PROBLEMS

1. Prove that $P_{2n}(0) = \dfrac{(-1)^n\,(2n)!}{2^{2n}\,(n!)^2}$.

(JNTU 2005, 2006 April, 2006 Nov., 2008 April/May)

Solution: The generating function for $P_n(x)$ is $(1 - 2xz + z^2)^{-1/2}$

$$\therefore \quad (1 - 2xz + z^2)^{-1/2} = \sum_0^\infty P_n(x)z^n$$

Substitute $x = 0$

$$(1 + z^2)^{-1/2} = \sum_0^\infty P_n(0)z^n$$

$$P_0(0) + P_1(0)z + P_2(0)z^2 + \cdots P_{2n}(0)z^{2n} + \cdots$$

$$= \left(1 - \frac{1}{2}z^2 + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}z^4 - \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{1}{6}z^6\right.$$

$$\left. + \cdots + (-1)^n \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2n-1}{2}\cdot\frac{1}{n!}z^{2n} + \cdots\right)$$

$$P_{2n}(0) = \frac{(-1)^n 1\cdot 3\cdot 5 \ldots 2n-1}{2^n}\frac{1}{n!}$$

$$= \frac{(-1)^n\,1\cdot 2\cdot 3\cdot 4 \ldots 2n}{2^n\,2\cdot 4\cdot 6 \ldots 2n \cdot n!}$$

[Multiplying and Dividing by $2 \cdot 4 \cdot 6 \cdots 2n$]

$$= \frac{(-1)^n \cdot (2n)!}{2^{2n} \cdot (n!)^2}$$

$$\therefore \quad P_{2n}(0) = \frac{(-1)^n\,2n!}{2^{2n}\,n!^2}.$$

2. Prove that $P_n(-x) = (-1)^n P_n(x)$. **(JNTU 2008 Aug./Sep.)**

Solution: We know that $(1 - 2xz + z^2)^{-1/2} = \sum_0^\infty P_n(x)z^n$ (i)

Substitute $x = -x$

$$(1 + 2xz + z^2)^{-1/2} = \sum_0^\infty P_n(-x)z^n$$

Substituting $z = -z$ in (i) (ii)

$$(1 + 2xz + z^2)^{-1/2} = \sum_0^\infty P_n(x)(-1)^n z^n$$ (iii)

Comparing (ii) and (iii)

$$\sum_0^\infty P_n(-x)z^n = \sum_0^\infty P_n(x)(-1)^n z^n$$

$$\therefore \quad P_n(-x) = (-1)^n P_n(x).$$

3. Prove that $P'_{n+1} + P_n' = P_0 + 3P_1 + 5P_2 + \cdots + (2n+1)P_n$. **(2006 April)**

Solution: $(2n+1)P_n = P'_{n+1} - P'_{n-1}$ recurrence formula

Substituting $n = 1, 2, 3 \ldots$

$$3P_1 = P_2' - P_0'$$

$$5P_2 = P_3' - P_1'$$

$$\vdots$$

$$(2n-1)P_{n-1} = P'_n - P'_{n-2}$$

$$(2n+1)P_n = P'_{n+1} - P'_{n-1}$$

Adding all

$3P_1 + 5P_2 + \cdots (2n-1)P_{n-1} + (2n+1)P_n$

$$= -P_0' + P_2' + P_3' - P_1' + P_5' - P_3' + \cdots P_n' - P'_{n-2} + P'_{n+1} - P'_{n-1}$$

Adding P_0

$$P_0 + 3P_1 + 5P_2 + 7P_3 + \cdots + (2n+1)P_n$$

$$= P_1' + P_2' + P_3' - P_1' + \cdots + P_n' - P'_{n-2} + P'_{n+1} - P'_{n-1}$$

$$(\because \quad P_0 = 1, P_0' = 0, P_1' = P_0)$$

$$= P'_{n+1} - P_n$$

$$\therefore \quad P'_{n+1} - P_n' = P_0 + 3P_1 + 5P_2 + \cdots + (2n+1)P_n.$$

4. Prove that $\int_{-1}^{1} P_n(x)(1-2xz+z^2)^{-1/2}dx = \dfrac{2z^n}{(2n+1)}$.

Solution: We know that

$$(1 - 2xz + z^2)^{-1/2} = \sum_0^\infty P_n(x)z^n$$

$$= P_0 + P_1 z + P_2 z^2 + \cdots + P_n z^n + \cdots$$

Multiplying both sides by $P_n(x)$ and integrating between -1 and 1 with respect to x.

$$\int_{-1}^{1} P_n(x)(1-2xz+z^2)^{-1/2}dx = \int_{-1}^{1} P_n(x)(P_0 + P_1 z + P_2 z^2 + \cdots + P_n z^n + \cdots)dx$$

$$\int_{-1}^{1} P_n(x)P_m(x)dx = 0 \quad (\text{if } n \neq m)$$

$$= \frac{2}{2n+1} \quad (\text{if } n = m)$$

$\therefore$
$$\int_{-1}^{1} P_n(x)P_0(x)dx = \int_{-1}^{1} P_n(x)P_1(x)dx$$

$$= \int_{-1}^{1} P_n(x)P_{n-1}(x)dx$$

$$= \int_{-1}^{1} P_n(x)P_{n+1}(x)dx$$

$$= 0$$

$\therefore$
$$\int_{-1}^{1} P_n(x)(1-2xz+z^2)^{-1/2}dx = \int_{-1}^{1} P_n(x)^2 z^n dx$$

$$= \frac{2z^n}{(2n+1)}$$

$\therefore$
$$\int_{-1}^{1} P_n(x)(1-2xz+z^2)^{-1/2}dx = \frac{2z^n}{(2n+1)}.$$

5. Prove that $\int_{-1}^{1} x^2 P_{n+1}P_{n-1}dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}$.

Solution: Consider recurrence formula

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}(x) \qquad \text{(i)}$$

Substitute $n \to n+1$ and $n-1$ in (i)

$$x(2n+3)P_{n+1}(x) = (n+2)P_{n+2}(x) + (n+1)P_n(x) \qquad \text{(ii)}$$

$$x(2n-1)P_{n-1} = nP_n + (n-1)P_{n-2} \qquad \text{(iii)}$$

Multiplying (ii) and (iii) and integrating with respect to x between -1 and 1

$$\int_{-1}^{1}(2n-1)(2n+3)x^2 P_{n+1}P_{n-1}dx = n(n+2)\int_{-1}^{1} P_n P_{n+2}dx + n(n+1)\int_{-1}^{1} P_n^{\,2}dx$$

$$+ (n-1)(n+2)\int_{-1}^{1} P_{n-2}P_{n+2}dx + (n^2-1)\int_{-1}^{1} P_n P_{n-2}dx$$

$$= n(n+1)\int_{-1}^{1} P_n^{\,2}dx \qquad \left[\because \int_{-1}^{1} P_m P_n dx = 0\right]$$

$$= \frac{2n(n+1)}{(2n+1)} \qquad [\text{if } m \neq n]$$

$$\int_{-1}^{1} P_n^2\,dx = \frac{2}{2n+1}$$

$\therefore$ $$\int_{-1}^{1} x^2 P_{n+1}P_{n-1}\,dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}.$$

6. Prove that $\int_{-1}^{1}(1-x^2)P_n'^2\,dx = \frac{2n(n+1)}{(2n+1)}$.

Solution: $\int_{-1}^{1}(1-x^2)\,P_n'^2dx = [(1-x^2)P_n'P_n]_{-1}^{1} - \int_{-1}^{1}P_n\frac{d}{dx}[(1-x^2)P_n']dx$

[Integrating by parts taking $(1-x^2)P_n'$ as first term]

$$= 0 - \int_{-1}^{1}P_n\frac{d}{dx}[(1-x^2)P_n']dx \qquad (i)$$

P_n satisfies the differential equation

$$\frac{d}{dx}[(1-x^2)P_n'] + n(n+1)P_n = 0$$

$\therefore$ $$\frac{d}{dx}[(1-x^2)P_n'] = -n(n+1)P_n$$

Substituting this value in (i)

$$\int_{-1}^{1}(1-x^2)P_n'^2\,dx = \int_{-1}^{1}n(n+1)P_n^2\,dx$$

$$= \frac{2n(n+1)}{(2n+1)}. \qquad \left[\because \int_{-1}^{1}P_n^2dx = \frac{2}{2n+1}\right]$$

7. Prove that (a) $P_n'(1) = \frac{1}{2}n(n+1)$; (b) $P_n'(-1) = (-1)^{n-1}\frac{n(n+1)}{2}$.

Solution: We know that P_n satisfies the differential equation

(a) $$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$\therefore$ $$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \qquad (i)$$

Put $x = 1$

$$-2P_n'(1) + n(n+1)P_n(1) = 0$$

$\therefore$ $$P_n'(1) = \frac{n(n+1)}{2} \qquad [\because P_n(1) = 1]$$

(b) Substitute $x = -1$ in (i)

$$2P_n'(-1) + n(n+1)P_n(-1) = 0$$

But

$$P_n(-x) = (-1)^n P_n(x)$$

$\therefore$ $$P_n(-1) = (-1)^n P_n(1) = (-1)^n$$

$\therefore$ $$2P_n'(-1) + n(n+1)(-1)^n = 0$$

$\therefore$ $$P_n'(-1) = (-1)^{n-1}\frac{n(n+1)}{2}.$$

8. Prove that $\dfrac{1+z}{z\sqrt{1-2xz+z^2}}-\dfrac{1}{z}=\sum_0^\infty[P_n(x)+P_{n+1}(x)]z^n$.

Solution:

$$\frac{1+z}{z\sqrt{1-2xz+z^2}}-\frac{1}{z}=\frac{1}{z}(1-2xz+z^2)^{-1/2}+(1-2xz+z^2)^{-1/2}-\frac{1}{z}$$

$$=\frac{1}{z}\sum_0^\infty P_n(x)z^n+\sum_0^\infty P_n(x)z^n-\frac{1}{z}$$

$$=\frac{1}{z}\left[\sum_0^\infty P_n(x)z^n-1\right]+\sum_0^\infty P_n(x)z^n$$

$$=\frac{1}{z}[P_0+P_1z+P_2z^2+P_3z^3+\cdots-P_0]+\sum_0^\infty P_n(x)z^n$$

$$(\because\ P_0=1)$$

$$=\frac{1}{z}[P_1z+P_2z^2+P_3z^3+\cdots]+\sum_0^\infty P_n(x)z^n$$

$$=[P_1+P_2z+P_3z^2+\cdots+P_{n+1}z^n+\cdots]+\sum_0^\infty P_n(x)z^n$$

$$=\sum_0^\infty P_{n+1}z^n+\sum_0^\infty P_n(x)z^n$$

$$=\sum_0^\infty[P_{n+1}(x)+P_n(x)]z^n.$$

9. Prove that $P_n\left(\frac{-1}{2}\right)=P_0\left(\frac{-1}{2}\right)P_{2n}\left(\frac{1}{2}\right)+P_1\left(\frac{-1}{2}\right)P_{2n-1}\left(\frac{-1}{2}\right)+\cdots$

$$+P_{2n}\left(\frac{-1}{2}\right)P_0\left(\frac{1}{2}\right).$$

Solution: We know that

$$(1-2xz+z^2)^{-1/2}=\sum_0^\infty P_n(x)z^n \qquad \text{(i)}$$

Substitute $x=\dfrac{1}{2}$ in (i)

$$(1-z+z^2)^{-1/2}=\sum_0^\infty P_n\left(\frac{1}{2}\right)z^n \qquad \text{(ii)}$$

Substitute $x = \frac{-1}{2}$ in (i)

$$(1 + z + z^2)^{-1/2} = \sum_0^\infty P_n\left(\frac{-1}{2}\right)z^n \quad \text{(iii)}$$

Substitute $z = z^2$ in (iii)

$$(1 + z^2 + z^4)^{-1/2} = \sum_0^\infty P_n\left(\frac{-1}{2}\right)z^{2n} \quad \text{(iv)}$$

$$(1 + z^2 + z^4)^{-1/2} = [(1 + z^2)^2 - z^2]^{-1/2}$$

$$= (1 + z + z^2)^{-1/2}\,(1 - z + z^2)^{-1/2} \quad \text{(v)}$$

From (ii), (iii), (iv) and (v)

$$\sum_0^\infty P_n\left(\frac{-1}{2}\right)z^{2n} = \sum_0^\infty P_n\left(\frac{-1}{2}\right)z^n \cdot \sum_0^\infty P_n\left(\frac{1}{2}\right)z^n$$

$$= \left[P_0\left(\frac{-1}{2}\right) + P_1\left(\frac{-1}{2}\right)z + P_2\left(\frac{-1}{2}\right)z^2 + \cdots\right]$$

$$= \left[P_0\left(\frac{1}{2}\right) + P_1\left(\frac{1}{2}\right)z + P_2\left(\frac{1}{2}\right)z^2 + \cdots\right]$$

Comparing the coefficient of z^{2n} on both sides

$$P_n\left(\frac{-1}{2}\right) = P_0\left(\frac{-1}{2}\right) P_{2n}\left(\frac{1}{2}\right) + P_1\left(\frac{-1}{2}\right) P_{2n-1}\left(\frac{1}{2}\right)$$

$$+ P_2\left(\frac{-1}{2}\right) P_{2n-1}\left(\frac{1}{2}\right) + \cdots P_{2n}\left(\frac{-1}{2}\right) P_0\left(\frac{1}{2}\right)$$

10. Show that if $m < n$ $\int_{-1}^{1} x^m P_n(x)dx = 0$.

(JNTU 2005 April, 2008 Aug./Sep.)

Solution: By Rodrigue's formula

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

Multiplying both sides by x^m and integrating between -1 and 1 with respect to x

$$\int_{-1}^{1} x^m P_n(x)dx = \int_{-1}^{1}\frac{x^m}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n dx$$

Integrating by parts

$$\frac{1}{2^n n!}\left[x^m \frac{d^{n-1}}{dx^{n-1}}(x^2 - 1)^n\right]_{-1}^{1} - \int_{-1}^{1} mx^{m-1}\frac{d^{n-1}(x^2 - 1)^n}{dx^{n-1}}dx$$

$$= \frac{-m}{2^n n!}\int_{-1}^{1} x^{m-1}\frac{d^{n-1}}{dx^{n-1}}(x^2 - 1)^n dx \quad \left[\because \frac{d^{n-1}}{dx^n}(x^2 - 1)^n\Big|_{-1}^{1} = 0\right]$$

Integrating again

$$= \frac{(-1)^2 m(m-1)}{2^n n!} \int_{-1}^{1} x^{m-2} \frac{d^{n-2}}{dx^{n-2}} (x^2-1)^n dx$$

Integrating m times

$$= \frac{(-1)^m m!}{2^n n!} \int_{-1}^{1} \frac{d^{n-m}}{dx^{n-m}} (x^2-1)^n dx$$

$$= \frac{(-1)^m m!}{2^n \cdot n!} \frac{d^{n-m-1}}{dx^{n-m-1}} (x^2-1)^n \Bigg|_{-1}^{1} = 0.$$

11. Prove that $\int_{-1}^{1} x P_n(x) P_{n-1}(x) = \frac{2n}{(4n^2-1)}$.

(JNTU 2007 April, 2008 April/May)

Solution: Recurrence relation is

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

Multiplying by $P_{n-1}(x)$ and integrating with respect to x between $x = -1$ to 1

$$\int_{-1}^{1} (2n+1)\, x\, P_n(x) P_{n-1}(x) = (n+1)\int_{-1}^{1} P_{n+1}(x) P_{n-1}(x) + n\int_{-1}^{1} [P_{n-1}(x)]^2 dx$$

$$= \frac{n \cdot 2}{(2n-2+1)} \qquad \left[\begin{array}{l} \int_{-1}^{1} [P_n(x)]^2 dx = \frac{2}{(2n+1)} \\ \int_{-1}^{1} P_n(x) P_m(x) dx = 0 \end{array} \right]$$

$$= \frac{2n}{(2n-1)}$$

$$\therefore \qquad \int_{-1}^{1} x\, P_n(x)\, P_{n-1}(x) = \frac{2n}{(4n^2-1)}.$$